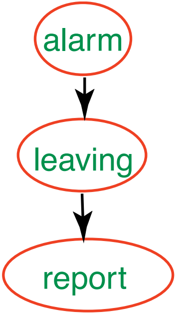
Reasoning Under Uncertainty: Variable Elimination

CPSC 322 – Uncertainty 6

Textbook §10.4

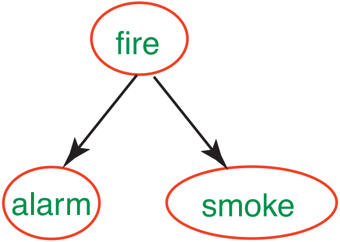
Lecture Overview

Chain



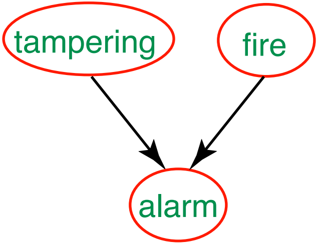
- *alarm* and *report* are independent: false.
- *alarm* and *report* are independent given *leaving*: true.
- Intuitively, the only way that the *alarm* affects *report* is by affecting *leaving*.

Common ancestors



- $alarm$ and $smoke$ are independent: false.
- $alarm$ and $smoke$ are independent given $fire$: true.
- Intuitively, $fire$ can explain $alarm$ and $smoke$; learning one can affect the other by changing your belief in $fire$.

Common descendants



- $tampering$ and $fire$ are independent: true.
- $tampering$ and $fire$ are independent given $alarm$: false.
- Intuitively, $tampering$ can explain away $fire$

Belief Network Inference

- Our goal: compute probabilities of variables in a belief network
- Two cases:
  1. the unconditional (prior) distribution over one or more variables
  2. the posterior distribution over one or more variables, conditioned on one or more observed variables
- To address both cases, we only need a computational solution to case 1
- Our method: exploiting the structure of the network to efficiently eliminate (sum out) the non-observed, non-query variables one at a time.

Factors

- A factor is a representation of a function from a tuple of random variables into a number.
    - denotes a distribution over the given tuple of variables in some (unspecified) context
    - Write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$
- We defined three operations on factors:
    1. Assigning one or more variables
        - $f(X_1 = v_1, X_2, \ldots, X_j)$ is a factor on $X_2, \ldots, X_j$, also written as $f(X_1, \ldots, X_j)_{X_1 = v_1}$
    2. Summing out variables
        - $(\sum_{X_1} f)(X_2, \ldots, X_j) =$ $f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)$
    3. Multiplying factors
        - $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) = f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z})$

Lecture Overview

1. Recap
2. Variable Elimination
3. Variable Elimination Example

Probability of a conjunction

- Suppose the variables of the belief network are $X_1, \ldots, X_n$.
- What we want to compute: the factor $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$
- We can compute $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$ by summing out the variables[1] $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} \setminus \{Z, Y_1, \ldots, Y_j\}$.
- We sum out these variables one at a time
  - the order in which we do this is called our elimination ordering.

$$P(Z, Y_1 = v_1, \ldots, Y_j = v_j) = \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}.$$

[1]Recall: Z_i and Y_i are alternate names for the variables from the set X, used to make indexing easier.

Probability of a conjunction

- What we know: the factors $P(X_i|pX_i)$.
- Using the chain rule and the definition of a belief network, we can write $P(X_1, \ldots, X_n)$ as $\prod_{i=1}^{n} P(X_i|pX_i)$. Thus:

$$P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$$
$$= \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}.$$
$$= \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)_{Y_1 = v_1, \ldots, Y_j = v_j}.$$

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?
  - factor out the a and then the h giving $a(b + c + d + h(e + f + g))$
  - this takes only 7 multiplications or additions

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?
  - factor out the a and then the h giving $a(b + c + d + h(e + f + g))$
  - this takes only 7 multiplications or additions
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)$ efficiently?

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?
  - factor out the a and then the h giving $a(b + c + d + h(e + f + g))$
  - this takes only 7 multiplications or additions

- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)$ efficiently?
- Factor out those terms that don't involve Z_1:

$$\left(\prod_{i|Z_1 \notin \{X_i\} \cup pX_i} P(X_i|pX_i) \right) \left(\sum_{Z_1} \prod_{i|Z_1 \in \{X_i\} \cup pX_i} P(X_i|pX_i) \right)$$

(terms that do not involve Z_i) *(terms that involve Z_i)*

Summing out a variable efficiently

To sum out a variable Z_j from a product $f_1, \dots, f_k$ of factors:

- Partition the factors into
  - those that don't contain Z_j, say $f_1, \dots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \dots, f_k$

We know:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = (f_1 \times \cdots \times f_i) \left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right).$$

- $\left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k\right)$ is a new factor; let's call it f'.
- Now we have:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f'.$$

- Store f' explicitly, and discard $f_{i+1}, \dots, f_k$. Now we've summed out Z_j.

Variable elimination algorithm

To compute $P(Q|Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

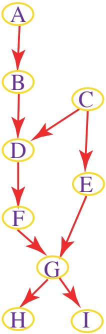
- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- For each of the other variables $Z_i \in \{Z_1, \ldots, Z_k\}$, sum out Z_i
- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(Q)$ by $\sum_Q f(Q)$.

Lecture Overview

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

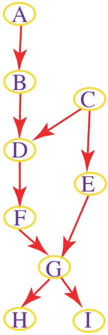
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A, B, C, D, E, F, G, H, I)$
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B, C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F, E) \cdot P(H|G) \cdot P(I|G)$



Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate A: $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$

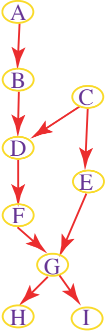


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate C: $P(G,H) =$
  $\sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$



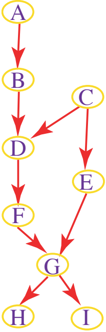
- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate E:

  $P(G,H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$

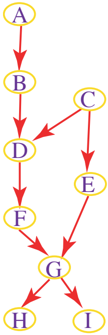


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$
- Observe $H = h_1$:
  $P(G, H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$



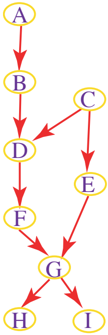
- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$
- Eliminate I:

  $P(G,H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$

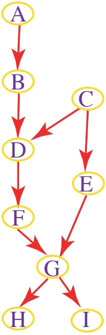


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H = h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B, D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate B:
  $P(G, H = h_1) = \sum_{D,F} f_6(D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$

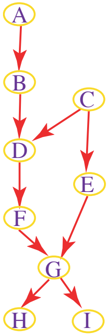


- $f_1(B) := \sum_{a \in dom(A)} P(A = a) \cdot P(B|A = a)$
- $f_2(B, D, E) := \sum_{c \in dom(C)} P(C = c) \cdot P(D|B, C = c) \cdot P(E|C = c)$
- $f_3(B, D, F, G) := \sum_{e \in dom(E)} f_2(B, D, E = e) \cdot P(G|F, E = e)$
- $f_4(G) := P(H = h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I = i|G)$
- $f_6(D, F, G) := \sum_{b \in dom(B)} f_1(B = b) \cdot f_3(B = b, D, F, G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{D,F} f_6(D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate D: $P(G, H=h_1) = \sum_{F} f_7(F,G) \cdot f_4(G) \cdot f_5(G)$

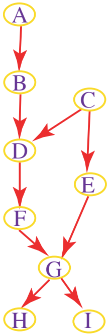


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_F f_7(F,G) \cdot f_4(G) \cdot f_5(G)$
- Eliminate F: $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$

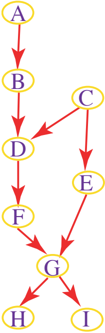


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$
- $f_8(G) := \sum_{f \in dom(F)} f_7(F=f,G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$
- Normalize: $P(G|H=h_1) = \frac{P(G,H=h_1)}{\sum_{g \in dom(G)} P(G,H=h_1)}$



- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B, D, E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B, C=c) \cdot P(E|C=c)$
- $f_3(B, D, F, G) := \sum_{e \in dom(E)} f_2(B, D, E=e) \cdot P(G|F, E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D, F, G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b, D, F, G)$
- $f_7(F, G) := \sum_{d \in dom(D)} f_6(D=d, F, G) \cdot P(F|D=d)$
- $f_8(G) := \sum_{f \in dom(F)} f_7(F=f, G)$