Reasoning Under Uncertainty: Variable Elimination Example

CPSC 322 Lecture 29

March 26, 2007
Textbook §9.5

Lecture Overview

1. Recap
2. More Variable Elimination
3. Variable Elimination Example

Summing out variables

Our second operation: we can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$
\begin{aligned}
&(\sum_{X_1} f)(X_2, \ldots, X_j) \\
&\quad = \quad f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)
\end{aligned}
$$

Multiplying factors

- Our third operation: factors can be multiplied together.
- The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Probability of a conjunction

- Suppose the variables of the belief network are $X_1, \ldots, X_n$.
- $\{X_1, \ldots, X_n\} = \{Q\} \cup \{Y_1, \ldots, Y_j\} \cup \{Z_1, \ldots,_k\}$
  - Q is the query variable
  - $\{Y_1, \ldots, Y_j\}$ is the set of observed variables
  - $\{Z_1, \ldots,_k\}$ is the rest of the variables, that are neither queried nor observed
- What we want to compute: $P(Q, Y_1 = v_1, \ldots, Y_j = v_j)$
- We can compute $P(Q, Y_1 = v_1, \ldots, Y_j = v_j)$ by summing out each variable $Z_1, \ldots, Z_k$
- We sum out these variables one at a time
  - the order in which we do this is called our elimination ordering.

$$P(Q, Y_1 = v_1, \ldots, Y_j = v_j) = \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}.$$

Probability of a conjunction

- What we know: the factors $P(X_i|pX_i)$.
- Using the chain rule and the definition of a belief network, we can write $P(X_1,\ldots,X_n)$ as $\prod_{i=1}^n P(X_i|pX_i)$. Thus:

$$
\begin{aligned}
& P(Q, Y_1 = v_1, \ldots, Y_j = v_j) \\
& \quad = \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
& \quad = \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?
  - factor out the a and then the h giving $a(b + c + d + h(e + f + g))$
  - this takes only 7 multiplications or additions
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)$ efficiently?
- Factor out those terms that don't involve Z_1:

$$\left(\prod_{i|Z_1 \notin \{X_i\} \cup pX_i} P(X_i|pX_i) \right) \left(\sum_{Z_1} \prod_{i|Z_1 \in \{X_i\} \cup pX_i} P(X_i|pX_i) \right)$$

(terms that do not involve Z_i) *(terms that involve Z_i)*

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Summing out a variable efficiently

To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

- Partition the factors into
  - those that don't contain Z_j, say $f_1, \ldots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \ldots, f_k$

We know:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = (f_1 \times \cdots \times f_i) \left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right).$$

- $\left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right)$ is a new factor; let's call it f'.
- Now we have:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f'.$$

- Store f' explicitly, and discard $f_{i+1}, \ldots, f_k$. Now we've summed out Z_j.

Variable elimination algorithm

To compute $P(Q|Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

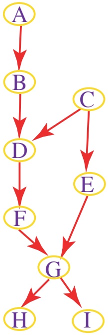
- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- For each of the other variables $Z_i \in \{Z_1, \ldots, Z_k\}$, sum out Z_i
- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(Q)$ by $\sum_Q f(Q)$.

Lecture Overview

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

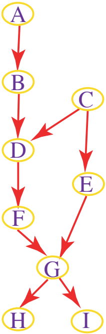
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A, B, C, D, E, F, G, H, I)$
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B, C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F, E) \cdot P(H|G) \cdot P(I|G)$



Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate A: $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$

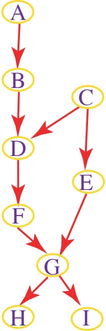


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate C: $P(G,H) =$
  $\sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$



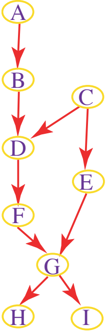
- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate E:

  $P(G,H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$



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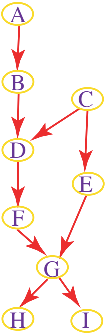
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$
- Observe $H=h_1$:

  $P(G,H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$



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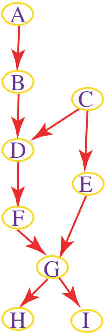
- $f_4(G) := P(H=h_1|G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$
- Eliminate I:

  $P(G, H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$



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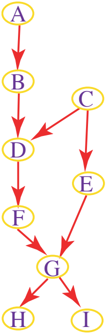
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B, D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate B:
  $P(G, H=h_1) = \sum_{D,F} f_6(D, F, G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$



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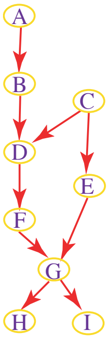
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D, F, G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b, D, F, G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{D,F} f_6(D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate D: $P(G, H=h_1) = \sum_{F} f_7(F,G) \cdot f_4(G) \cdot f_5(G)$



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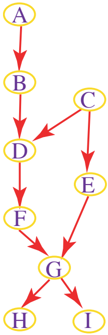
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_F f_7(F,G) \cdot f_4(G) \cdot f_5(G)$
- Eliminate F: $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$



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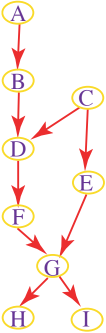
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$
- $f_8(G) := \sum_{f \in dom(F)} f_7(F=f,G)$

Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H = h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$
- Normalize: $P(G|H = h_1) = \frac{P(G, H = h_1)}{\sum_{g \in dom(G)} P(G, H = h_1)}$



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What good was Conditional Independence?

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- And... why did we bother learning conditional independence? Does it help us at all?

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- And... why did we bother learning conditional independence? Does it help us at all?
  - yes—we use the chain rule decomposition right at the beginning

- Can we use our knowledge of conditional independence to make this calculation even simpler?
  - yes—there are some variables that we don't have to sum out
  - intuitively, they're the ones that are "pre-summed-out" in our tables
  - example: summing out I on the previous slide

One Last Trick

One last trick to simplify calculations: we can repeatedly eliminate all leaf nodes that are neither observed nor queried, until we reach a fixed point.

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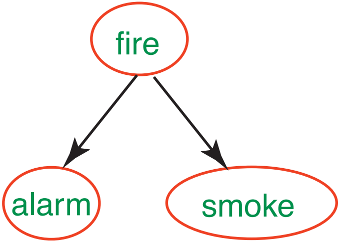

Can we justify that on a three-node graph—Fire, Alarm, and Smoke—when we ask for:

- $P(Fire)$?

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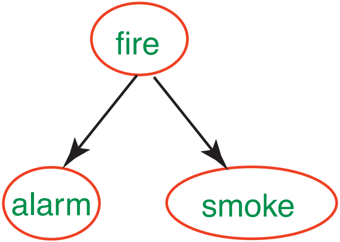

Can we justify that on a three-node graph—Fire, Alarm, and Smoke—when we ask for:

- $P(Fire)$?
- $P(Fire \mid Alarm)$?