CSPs: Arc Consistency

CPSC 322 Lecture 11

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Textbook §3.5, 3.8

Lecture Overview

1. Recap
2. Arc Consistency

CSPs as Search Problems

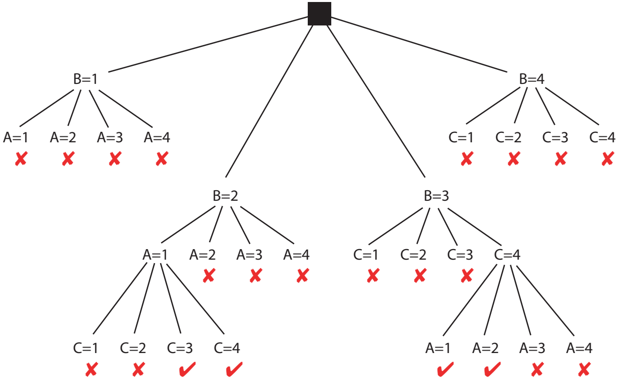
We map CSPs into search problems:

- nodes: assignments of values to a subset of the variables
- neighbours of a node: nodes in which values are assigned to one additional variable
- start node: the empty assignment (no variables assigned values)
- leaf node: a node which assigns a value to each variable
- goal node: leaf node which satisfies all of the constraints

Note: the path to a goal node is not important

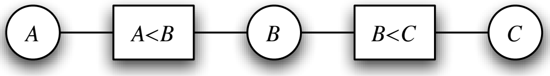
Example

An example of solving a CSP using depth-first search, with pruning whenever a partial assignment violates a constraint



Constraint Networks

- A constraint network:
  - Two kinds of nodes in the graph
    - one node for every variable
    - one node for every constraint
  - Edges run between variable nodes and constraint nodes: they indicate that a given variable is involved in a given constraint



Lecture Overview

Arc Consistency

Definition

An arc $\langle X, r(X, \bar{Y})\rangle$ is arc consistent if for each value of X in $\mathbf{D}_X$ there is some value $\bar{Y}$ in $\mathbf{D}_{\bar{Y}}$ such that $r(X, \bar{Y})$ is satisfied.

- In symbols, $\forall X \in \mathbf{D}_X, \ \exists \bar{Y} \in \mathbf{D}_{\bar{Y}}$ such that $r(X, \bar{Y})$ is satisfied.
- A network is arc consistent if all its arcs are arc consistent.
- If an arc $\langle X, \bar{Y}\rangle$ is *not* arc consistent, all values of X in $\mathbf{D}_X$ for which there is no corresponding value in $\mathbf{D}_{\bar{Y}}$ may be deleted from $\mathbf{D}_X$ to make the arc $\langle X, \bar{Y}\rangle$ consistent.
    - This removal can never rule out any models (do you see why?)

Arc Consistency Algorithm

- Consider the arcs in turn making each arc consistent.
  - An arc $\langle X, r(X, \bar{Y}) \rangle$ needs to be revisited if the domain of Y is reduced.
- Regardless of the order in which arcs are considered, we will terminate with the same result: an arc consistent network.
- Worst-case complexity of this procedure:
  - let the max size of a variable domain be d
  - let the number of constraints be e
  - complexity is $O(ed^3)$
- Some special cases are faster
  - e.g., if the constraint graph is a tree, arc consistency is $O(ed)$

Arc Consistency Algorithm

```
procedure AC(V, dom, R)
    Inputs
        V: a set of variables
        dom: a function such that dom(X) is the domain of variable X
        R: set of relations to be satisfied
    Output
        arc consistent domains for each variable
    Local
        D_X is a set of values for each variable X
    for each variable X do
        D_X ← dom(X)
    end for each
    TDA ← {⟨X, r⟩ | r ∈ R is a constraint that involves X}
    while TDA ≠ {} do
        select ⟨X, r⟩ ∈ TDA;
        TDA ← TDA − {⟨X, r⟩};
        ND_X ← {x | x ∈ D_X and there is ȳ ∈ D_Ȳ such that r(x, ȳ)};
        if ND_X ≠ D_X then
            TDA ← TDA ∪ {⟨Z, r′⟩ | r′ ≠ r and r′ involves X and Z ≠ X};
            D_X ← ND_X;
        end if
    end while
    return {D_X : X is a variable}
end procedure
```

Adding edges back to TDA

- When we change the domain of a variable X in the course of making an arc $\langle X, r \rangle$ arc consistent, we add every arc $\langle Z, r' \rangle$ where r' involves X and:
  - $r \neq r'$
  - $Z \neq X$
- Thus we don't add back the same arc:
  - This makes sense—it's definitely arc consistent.

Adding edges back to TDA

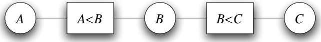
- When we change the domain of a variable X in the course of making an arc $\langle X, r\rangle$ arc consistent, we add every arc $\langle Z, r'\rangle$ where r' involves X and:
  - $r \neq r'$
  - $Z \neq X$

- We don't add back other arcs that involve the same variable X
  - We've just *reduced* the domain of X
  - If an arc $\langle X, r\rangle$ was arc consistent before, it will still be arc consistent
    - in the "for all" we'll just check fewer values

Adding edges back to TDA

- When we change the domain of a variable X in the course of making an arc $\langle X, r\rangle$ arc consistent, we add every arc $\langle Z, r'\rangle$ where r' involves X and:
  - $r \neq r'$
  - $Z \neq X$
- We don't add back other arcs that involve the same constraint and a different variable:
  - Imagine that such an arc—involving variable Y—had been arc consistent before, but was no longer arc consistent after X's domain was reduced.
  - This means that some value in Y's domain could satisfy r only when X took one of the dropped values
  - But we dropped these values precisely because there were no values of Y that allowed r to be satisfied when X takes these values—contradiction!

Arc Consistency Example



- $dom(A) = \{1,2,3,4\}; dom(B) = \{1,2,3,4\}; dom(C) = \{1,2,3,4\}$
- Suppose you first select the arc $\langle A, A < B\rangle$.
  - Remove $A = 4$ from the domain of A.
  - Add nothing to TDA.
- Suppose that $\langle B, B < C\rangle$ is selected next.
  - Prune the value 4 from the domain of B.
  - Add $\langle A, A < B\rangle$ back into the TDA set (why?)
- Suppose that $\langle B, A < B\rangle$ is selected next.
  - Prune 1 from the domain of B.
  - Add no element to TDA (why?)
- Suppose the arc $\langle A, A < B\rangle$ is selected next
  - The value $A = 3$ can be pruned from the domain of A.
  - Add no element to TDA (why?)
- Select $\langle C, B < C\rangle$ next.
  - Remove 1 and 2 from the domain of C.
  - Add $\langle B, B < C\rangle$ back into the TDA set

The other two edges are arc consistent, so the algorithm terminates with $dom(A) = \{1,2\}$, $dom(B) = \{2,3\}$, $dom(C) = \{3,4\}$.

Arc Consistency Outcomes

- Three possible outcomes (when all arcs are arc consistent):
  - One domain is empty $\Rightarrow$ no solution
  - Each domain has a single value $\Rightarrow$ unique solution
  - Some domains have more than one value $\Rightarrow$ may or may not be a solution
    - in this case, arc consistency isn't enough to solve the problem: we need to perform search