Game Theory: Analyzing Games

CPSC 322 Lecture 35

April 5, 2006

Reading: excerpt from "Multiagent Systems", chapter 3.

Lecture Overview

Recap

Pareto Optimality

Best Response and Nash Equilibrium

Mixed Strategies

Non-Cooperative Game Theory

- What is it?
  - mathematical study of interaction between rational, self-interested agents
- Why is it called non-cooperative?
  - while it's most interested in situations where agents' interests conflict, it's not restricted to these settings
  - the key is that the individual is the basic modeling unit, and that individuals pursue their own interests
    - cooperative/coalitional game theory has teams as the central unit, rather than agents
- You can think of a non-cooperative game as a decision diagram where different agents control different decision nodes, and where each agent has his own utility node.

Defining Games

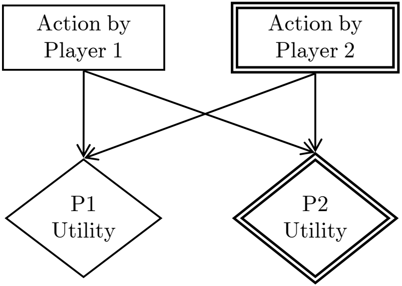
- Finite, n-person game: $\langle N, A, u \rangle$:
  - N is a finite set of n players, indexed by i
  - $A = A_1, \ldots, A_n$ is a set of actions for each player i
    - $a \in A$ is an action profile
  - $u = \{u_1, \ldots, u_n\}$, a utility function for each player, where $u_i : A \mapsto \mathbb{R}$

- Writing a 2-player game as a matrix:
  - row player is player 1, column player is player 2
  - rows are actions $a \in A_1$, columns are $a' \in A_2$
  - cells are outcomes, written as a tuple of utility values for each player

Games in Matrix Form

Here's the TCP Backoff Game written as a matrix ("normal form") and as a decision network.

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



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- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?

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- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - we have no way of saying that one agent's interests are more important than another's
  - intuition: imagine trying to find the revenue-maximizing outcome when you don't know what currency has been used to express each agent's payoff
- Are there situations where we can still prefer one outcome to another?

Pareto Optimality

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o', and there is no agent who strictly prefers o' to o
  - in this case, it seems reasonable to say that o is better than o'
  - we say that o Pareto-dominates o'.

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- An outcome o^* is Pareto-optimal if there is no other outcome which Pareto-dominates it.
  - can a game have more than one Pareto-optimal outcome?
  - does every game have at least one Pareto-optimal outcome?

Pareto Optimal Outcomes in Example Games

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F	0, 0	1, 2

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Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
  - phrased as a decision diagram: observing the other players' decision nodes as evidence
- Let $a_{-i} = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$.
  - now $a = (a_{-i}, a_i)$
- Best response: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i,\ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$

Nash Equilibrium

- Now let's return to the setting where no agent knows anything about what the others will do
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- What can we say about which actions will occur?

- Idea: look for stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$ is a Nash equilibrium iff $\forall i,\ a_i \in BR(a_{-i})$.

Nash Equilibria of Example Games

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The paradox of Prisoner's dilemma: the Nash equilibrium is the only non-Pareto-optimal outcome!

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- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i.
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

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- Nash equilibrium:
  - $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i,\ s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
  - e.g., matching pennies: both players play heads/tails 50%/50%