Reasoning Under Uncertainty: Introduction to Probability

CPSC 322 Lecture 23

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Textbook §9

Lecture Overview

Recap

Probability Introduction

Syntax and Semantics of Probability

Objects and Relations

- It is useful to view the world as consisting of objects and relationships between these objects.
- Often the propositions we spoke about before can be condensed into a much smaller number of propositions if they are allowed to express relationships between objects and/or functions of objects.
- Thus, reasoning in terms of objects and relationships can be simpler than reasoning in terms of features, as you can express more general knowledge using logical variables.

Syntax of Datalog

- variable starts with upper-case letter.
- constant starts with lower-case letter or is a sequence of digits (numeral).
- predicate symbol starts with lower-case letter.
- term is either a variable or a constant.
- atomic symbol (atom) is of the form p or $p(t_1, \ldots, t_n)$ where p is a predicate symbol and t_i are terms.

Syntax of Datalog (cont)

- definite clause is either an atomic symbol (a fact) or of the form:

$$\underbrace{a}_{\text{head}} \leftarrow \underbrace{b_1 \wedge \cdots \wedge b_m}_{\text{body}}$$

  where a and b_i are atomic symbols.
- query is of the form $?b_1 \wedge \cdots \wedge b_m$.
- knowledge base is a set of definite clauses.

Formal Semantics

An interpretation is a triple $I = \langle D, \phi, \pi \rangle$, where

- D, the domain, is a nonempty set. Elements of D are individuals.
- ϕ is a mapping that assigns to each constant an element of D. Constant c denotes individual $\phi(c)$.
- π is a mapping that assigns to each n-ary predicate symbol a relation: a function from D^n into $\{$*TRUE*, *FALSE*$\}$.

Truth in an interpretation

A constant c denotes in I the individual $\phi(c)$.
Ground (variable-free) atom $p(t_1, \ldots, t_n)$ is

- true in interpretation I if $\pi(p)(t'_1, \ldots, t'_n) =$ *TRUE*, where t_i denotes t'_i in interpretation I and
- false in interpretation I if $\pi(p)(t'_1, \ldots, t'_n) =$ *FALSE*.

Ground clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ is false in interpretation I if h is false in I and each b_i is true in I, and is true in interpretation I otherwise.

Variables

- Variables are universally quantified in the scope of a clause.
- A variable assignment is a function from variables into the domain.
- Given an interpretation and a variable assignment, each term denotes an individual and each clause is either true or false.
- A clause containing variables is true in an interpretation if it is true for all variable assignments.

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Using Uncertain Knowledge

- Agents don't have complete knowledge about the world.
- Agents need to make decisions based on their uncertainty.
- It isn't enough to assume what the world is like.
  Example: wearing a seat belt.
- An agent needs to reason about its uncertainty.
- When an agent makes an action under uncertainty, it is gambling $\Longrightarrow$ probability.

Probability

- Probability is formal measure of uncertainty. There are two camps:
- Frequentists: believe that probability represents something *objective*, and compute probabilities by counting the frequencies of different events
- Bayesians: believe that probability represents something *subjective*, and understand probabilities as degrees of belief.
  - They compute probabilities by starting with prior beliefs, and then updating beliefs when they get new data.
  - Example: Your degree of belief that a bird can fly is your measure of belief in the flying ability of an individual based only on the knowledge that the individual is a bird.
  - Other agents may have different probabilities, as they may have had different experiences with birds or different knowledge about this particular bird.
  - An agent's belief in a bird's flying ability is affected by what the agent knows about that bird.

Numerical Measures of Belief

- Belief in proposition, f, can be measured in terms of a number between 0 and 1 — this is the probability of f.
  - The probability f is 0 means that f is believed to be definitely false.
  - The probability f is 1 means that f is believed to be definitely true.
- Using 0 and 1 is purely a convention.
- f has a probability between 0 and 1, doesn't mean f is true to some degree, but means you are ignorant of its truth value. Probability is a measure of your ignorance.

Random Variables

- A random variable is a term in a language that can take one of a number of different values.
- The domain of a variable X, written $dom(X)$, is the set of values X can take.
- A tuple of random variables $\langle X_1, \ldots, X_n \rangle$ is a complex random variable with domain $dom(X_1) \times \cdots \times dom(X_n)$. Often the tuple is written as $X_1, \ldots, X_n$.
- Assignment $X = x$ means variable X has value x.
- A proposition is a Boolean formula made from assignments of values to variables.

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Possible World Semantics

- A possible world specifies an assignment of one value to each random variable.
- $w \models X = x$ means variable X is assigned value x in world w.
- Logical connectives have their standard meaning:

$$w \models \alpha \wedge \beta \text{ if } w \models \alpha \text{ and } w \models \beta$$
$$w \models \alpha \vee \beta \text{ if } w \models \alpha \text{ or } w \models \beta$$
$$w \models \neg\alpha \text{ if } w \not\models \alpha$$

- Let Ω be the set of all possible worlds.

Semantics of Probability: finite case

For a finite number of possible worlds:

- Define a nonnegative measure $\mu(w)$ to each world w so that the measures of the possible worlds sum to 1.
- The measure specifies how much you think the world w is like the real world.
- The probability of proposition f is defined by:

$$P(f) = \sum_{w \models f} \mu(w).$$

Axioms of Probability: finite case

Four axioms define what follows from a set of probabilities:

Axiom 1 $P(f) = P(g)$ if $f \leftrightarrow g$ is a tautology. That is, logically equivalent formulae have the same probability.

Axiom 2 $0 \leq P(f)$ for any formula f.

Axiom 3 $P(\tau) = 1$ if τ is a tautology.

Axiom 4 $P(f \vee g) = P(f) + P(g)$ if $\neg(f \wedge g)$ is a tautology.

- These axioms are sound and complete with respect to the semantics.

Semantics of Probability: general case

In the general case we have a measure on sets of possible worlds, satisfying:

- $\mu(S) \geq 0$ for $S \subseteq \Omega$
- $\mu(\Omega) = 1$
- $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$ if $S_1 \cap S_2 = \{\}$.