Comparing Forward and Backward Reachability as Tools for Safety Analysis

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research supported by
National Science and Engineering Research Council of Canada

Outline

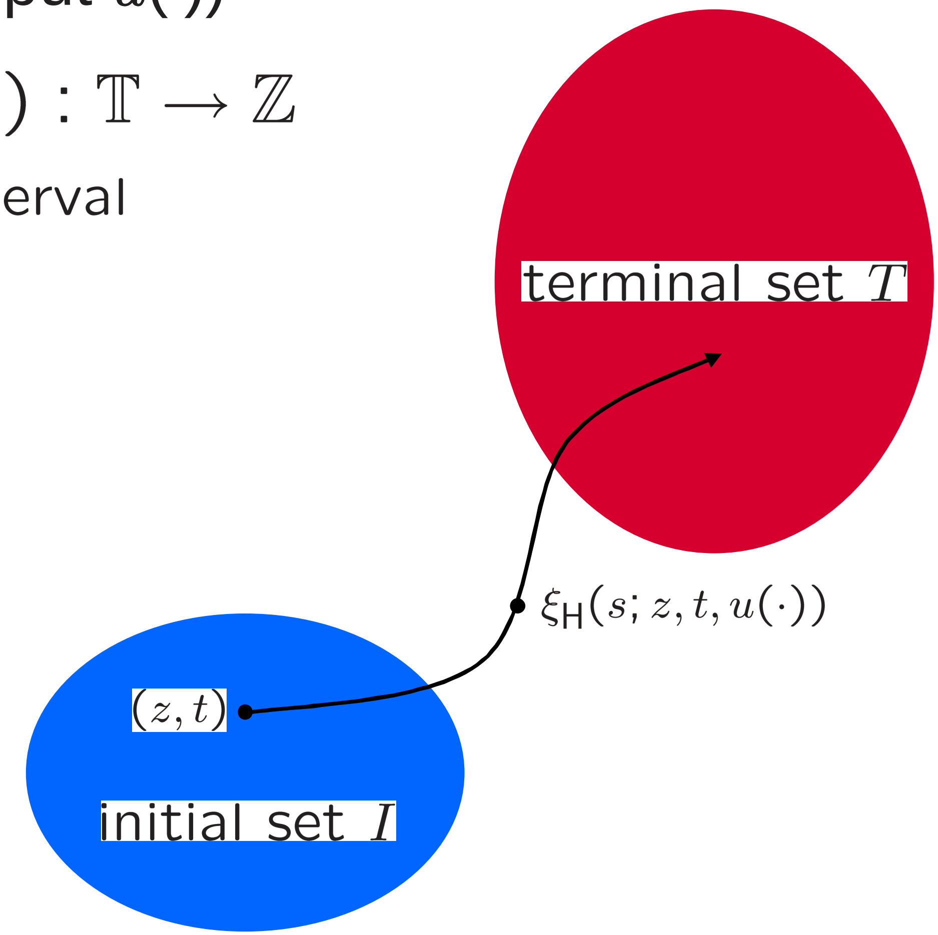
- Definitions
  - safety analysis and system models
  - forward and backward reach sets and tubes
- Exchanging algorithms by time reversal
- Safety analysis with different input policies
  - maximal reachability
  - minimal reachability
- Sensitivity of reachability operators
  - ill conditioned continuous & hybrid examples

Safety Analysis

- Does there exist a trajectory of system **H** leading from a state in initial set I to a state in terminal set T? (under some policy for input $u(\cdot)$)

Trajectory $\xi_{\mathsf{H}}(s; z, t, u(\cdot)) : \mathbb{T} \to \mathbb{Z}$

- $\mathbb{T} = [-\mathcal{T}, +\mathcal{T}]$ is time interval
- $\mathbb{Z}$ is state space of H
- $s \in \mathbb{T}$ is current time
- $z \in \mathbb{Z}$ is initial state
- $t \in \mathbb{T}$ is initial time
- $u(\cdot) \in \mathbb{U}$ is input signal



Assumption:

Given z, t and $u(\cdot)$ trajectory is unique

Typical Systems: ODEs

- Common model for continuous state spaces
- Standard existence and uniqueness

$$\dot{z}(t) = f(z(t), u(t))$$

- $f : \mathbb{Z} \times U \to \mathsf{T}\mathbb{Z}$ are dynamics
- $U \subset \mathbb{R}^{d_u}$ is convex and compact
- $\mathbb{U} = \{\phi : \mathbb{T} \to U \mid \phi(\cdot) \text{ is measurable}\}$
- Often $\mathbb{Z} \subseteq \mathbb{R}^{d_z}$

System specified by $\mathsf{H}_\mathsf{C} = (\mathbb{Z}, f, U)$

Typical Systems: Hybrid Automata

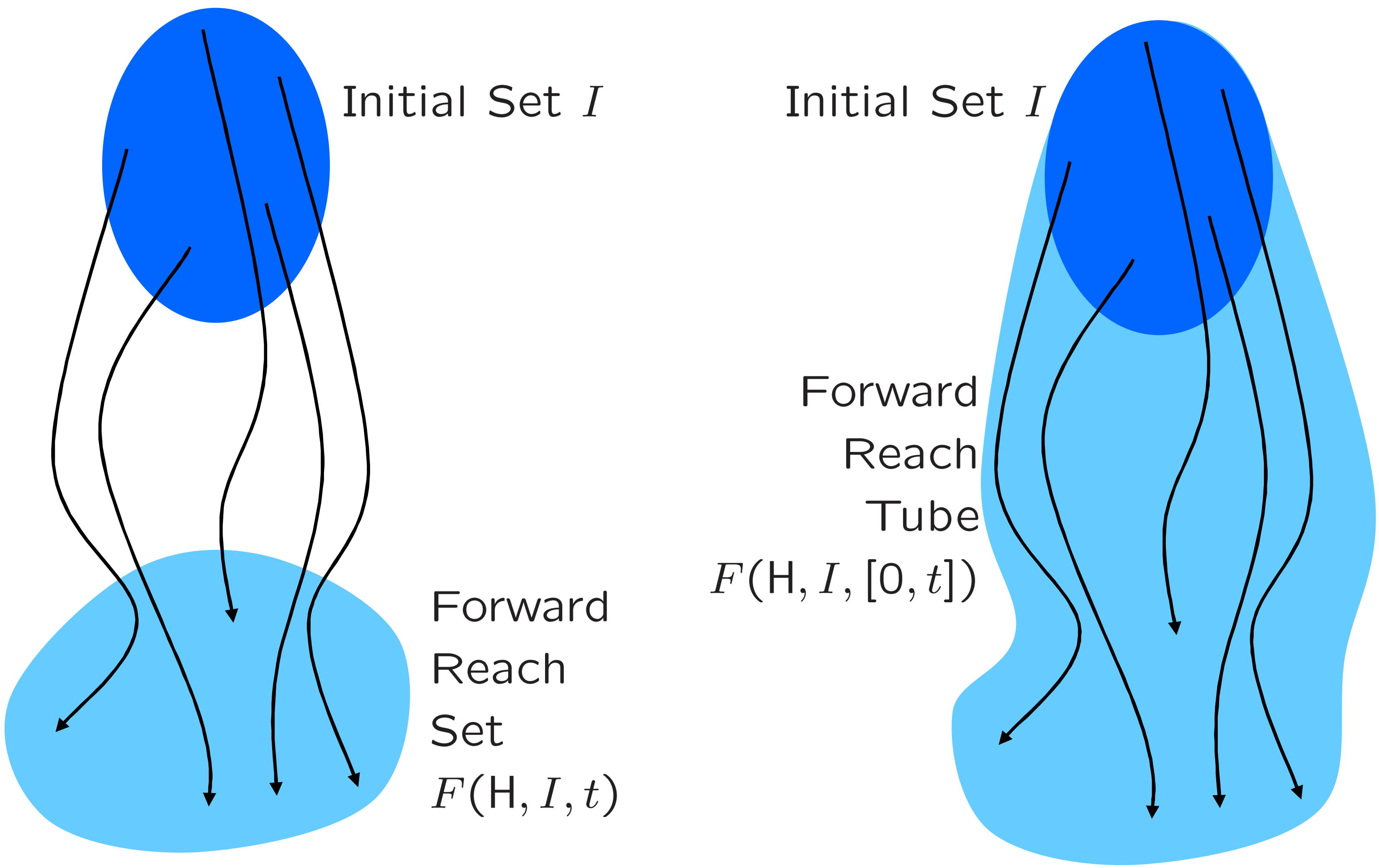
- Adapted from [Gao, Lygeros & Quincampoix 2006]
- Challenging existence and uniqueness
  - eg: [Broucke & Arapostathis, Sys. & Con. Letters 2002] or [Lygeros, Johansson, Simic, Zhang & Sastry, TAC 2003]
  - requires at least non-Zeno and non-blocking
  - all non-determinism must be expressed through input $u(\cdot)$

System specified by $\mathsf{H}_{\mathsf{H}} = (\mathbb{Q}, \mathbb{X}, f, D, G, r, U)$

$\mathbb{Q}$	discrete states;
$\mathbb{X}$	continuous states;
$f : \mathbb{Q} \times \mathbb{X} \times U_C \to \mathsf{T}\mathbb{X}$	continuous dynamics (vector field);
$D : \mathbb{Q} \times U_D \to P(\mathbb{X})$	domain of continuous evolution;
$G : \mathbb{Q} \times \mathbb{Q} \times U_D \to P(\mathbb{X})$	guards for discrete evolution;
$r : \mathbb{Q} \times \mathbb{Q} \times \mathbb{X} \times U \to \mathbb{X}$	reset function;
$U = (U_C, U_D)$	continuous and discrete input sets;

Forward Reachability

- Start at initial conditions and compute forward

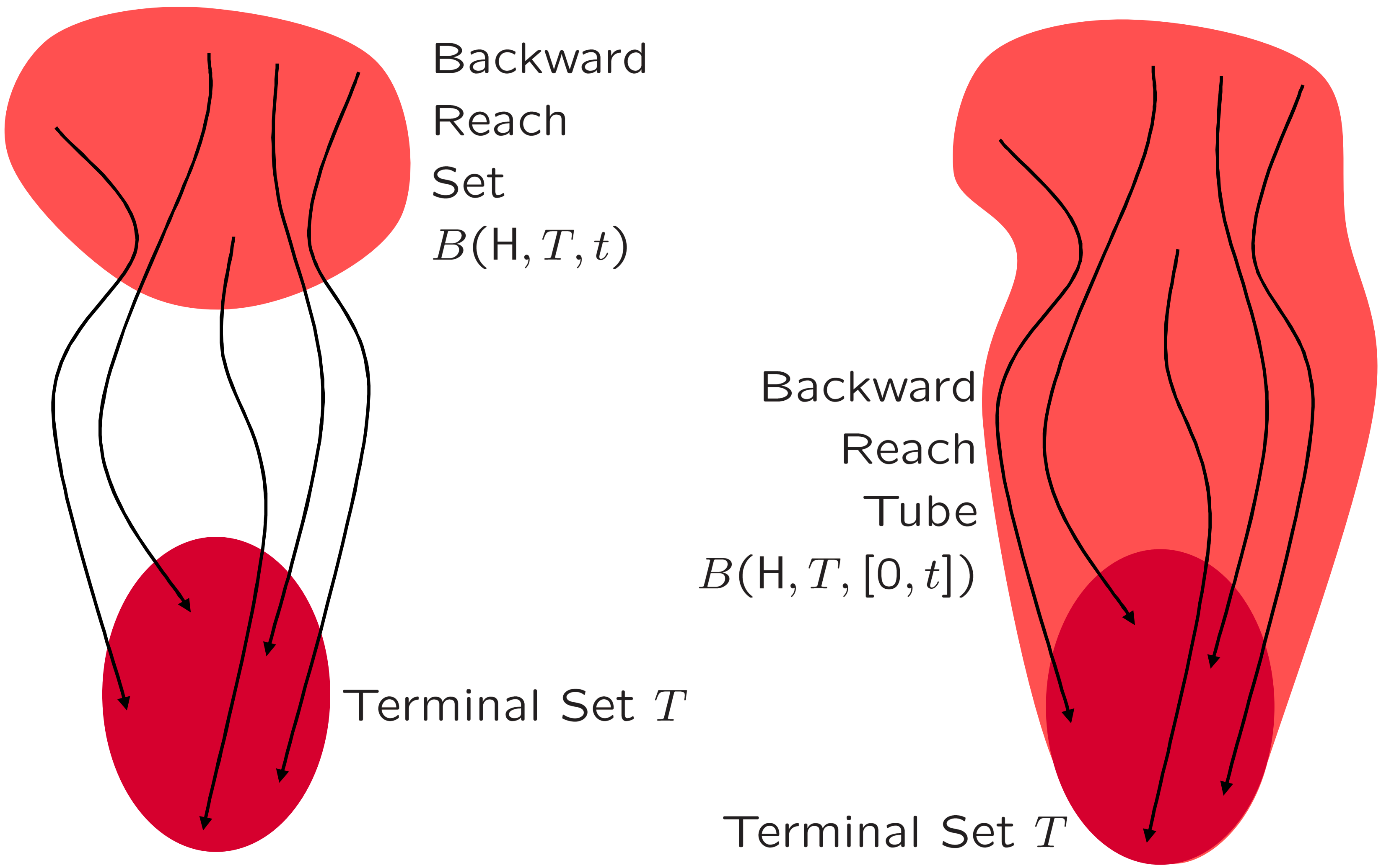


Forward Reachability Algorithms

- Forward approach typical of Lagrangian algorithms
  - Representation moves with the underlying dynamics
  - Varying ability to handle nonlinearity and/or inputs
  - Demonstrated ability to handle high dimensions
- Examples
  - [Henzinger, Ho & Wong-Toi, IEEE TAC 1998]
  - [Greenstreet & Mitchell, HSCC 1999]
  - [Bemporad, Torrisi & Morari HSCC 2000]
  - [Kurzhanski & Varaiya, HSCC 2000]
  - [Asarin, Dang & Girard, HSCC 2003]
  - [Girard, Guernic & Maler, HSCC 2006]
  - [Han & Krogh, HSCC 2006]

Backward Reachability

- Start at terminal set and compute backwards



Backward Reachability Algorithms

- Backward approach typical of Eulerian algorithms
  - Representation not moving (although it may adapt)
  - Generally handle nonlinear and multiple inputs
  - No examples beyond four dimensions?
- Examples
  - [Broucke, Benedetto, Gennaro & Sangiovanni-Vincentelli, HSCC 2001]
  - [Saint-Pierre, HSCC 2002]
  - [Sethian & Vladimirsky, HSCC 2002]
  - [Mitchell, Bayen & Tomlin, IEEE TAC 2005]
  - [Gao, Lygeros & Quincampoix, HSCC 2006]

Exchanging Algorithms

- Algorithms are (mathematically) interchangeable if system dynamics can be reversed in time

$$\text{Backward dynamic system } \overleftarrow{\mathsf{H}}$$
$$\text{such that } \forall s, t \in \mathbb{T}$$
$$\xi_{\mathsf{H}}(s; z, t, u(\cdot)) = \hat{z} \iff \xi_{\overleftarrow{\mathsf{H}}}(s; \hat{z}, t, u(\cdot)) = z.$$

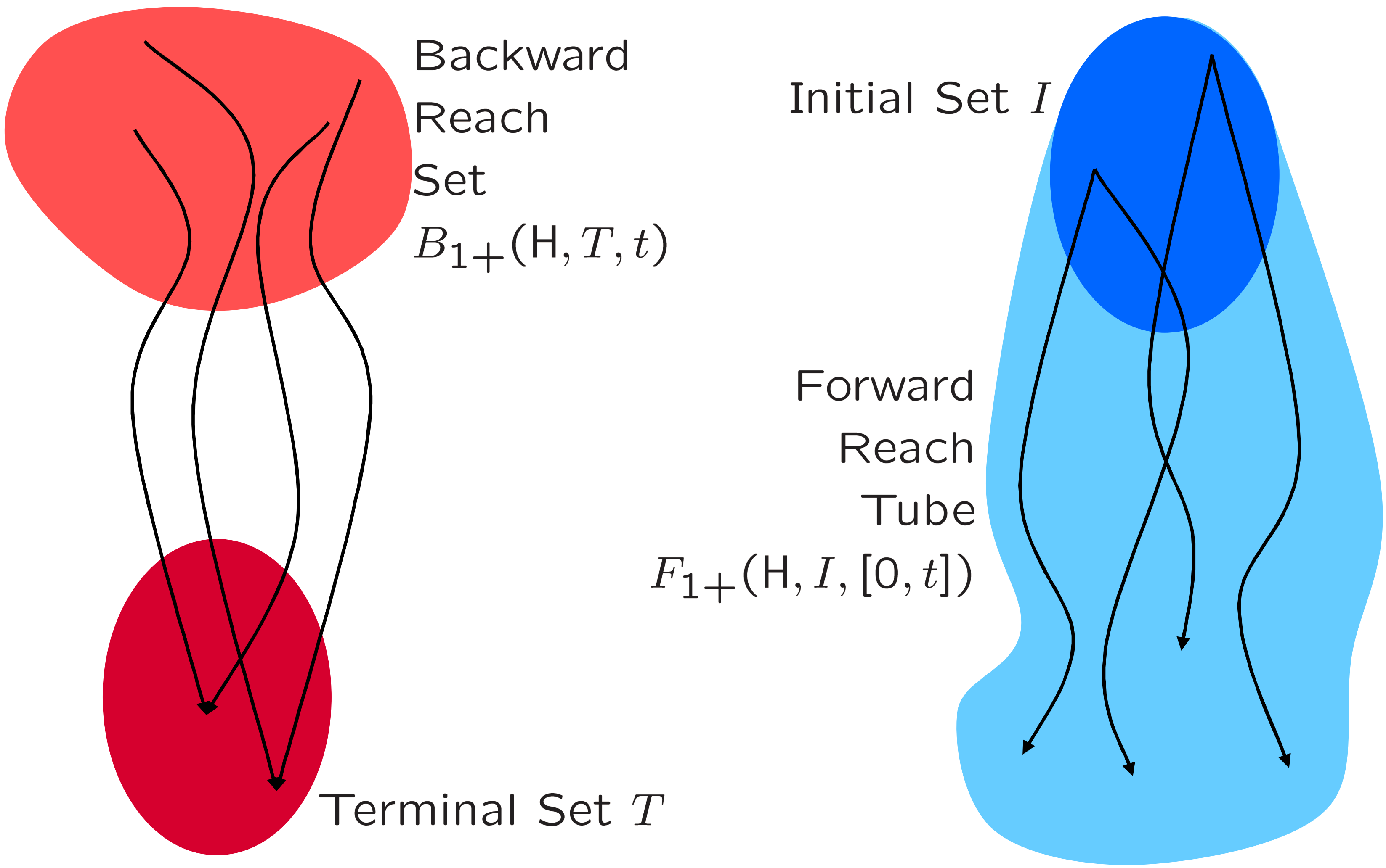
- For example: $\overleftarrow{\mathsf{H}_{\mathsf{C}}} = (\mathbb{Z}, -f, U)$
  $$\overleftarrow{\mathsf{H}_{\mathsf{H}}} = (\mathbb{Q}, \mathbb{X}, -f, D, \overleftarrow{G}, \overleftarrow{r}, U)$$

- Then

$$F(\mathsf{H}, S, [0, t]) = B(\overleftarrow{\mathsf{H}}, S, [0, t])$$
$$F(\mathsf{H}, S, t) = B(\overleftarrow{\mathsf{H}}, S, t)$$

Maximal Reachability

- Input signal $u(\cdot)$ maximizes size of the set or tube



Maximal Reachability Definition

$$F_{1+}(\mathsf{H}, S, t) \triangleq \{\hat{z} \in \mathbb{Z} \mid \exists u(\cdot) \in \mathbb{U}, \exists z \in S,$$
$$\xi_{\mathsf{H}}(t; z, 0, u(\cdot)) = \hat{z}\}$$
$$F_{1+}(\mathsf{H}, S, [0, t]) \triangleq \{\hat{z} \in \mathbb{Z} \mid \exists u(\cdot) \in \mathbb{U}, \exists z \in S, \exists s \in [0, t],$$
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$$\xi_{\mathsf{H}}(0; z, -s, u(\cdot)) = \hat{z}\}$$

Maximal results also apply to systems without input; for example: $F_0(\mathsf{H}, S, t) \triangleq \{\hat{z} \in \mathbb{Z} \mid \exists z \in S, \xi_{\mathsf{H}}(t; z, 0) = \hat{z}\}$.

Maximal Reachability Results

- Reach sets and tubes provide similar information

$$F_{1+}(\mathsf{H}, S, [0,t]) = \bigcup_{\hat{t} \in [0,t]} F_{1+}(\mathsf{H}, S, \hat{t})$$

$$B_{1+}(\mathsf{H}, S, [0,t]) = \bigcup_{\hat{t} \in [0,t]} B_{1+}(\mathsf{H}, S, \hat{t})$$

- The following properties are equivalent

1. H is safe over horizon $t \leq \mathcal{T}$ for all possible inputs $u(\cdot) \in \mathbb{U}$.
2. $F_{1+}(\mathsf{H}, I, s) \cap T = \emptyset$ for all $s \in [0, t]$.
3. $F_{1+}(\mathsf{H}, I, [0, t]) \cap T = \emptyset$.
4. $B_{1+}(\mathsf{H}, T, s) \cap I = \emptyset$ for all $s \in [0, t]$.
5. $B_{1+}(\mathsf{H}, T, [0, t]) \cap I = \emptyset$.

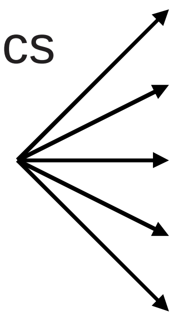
- Any maximal reachability operator can be used to demonstrate safety for all possible inputs

Maximal Reachability Demonstration

System Dynamics

$$\dot{x} = \begin{bmatrix} +1 \\ u \end{bmatrix}$$

$$|u| \leq 1$$



Initial and Terminal Sets

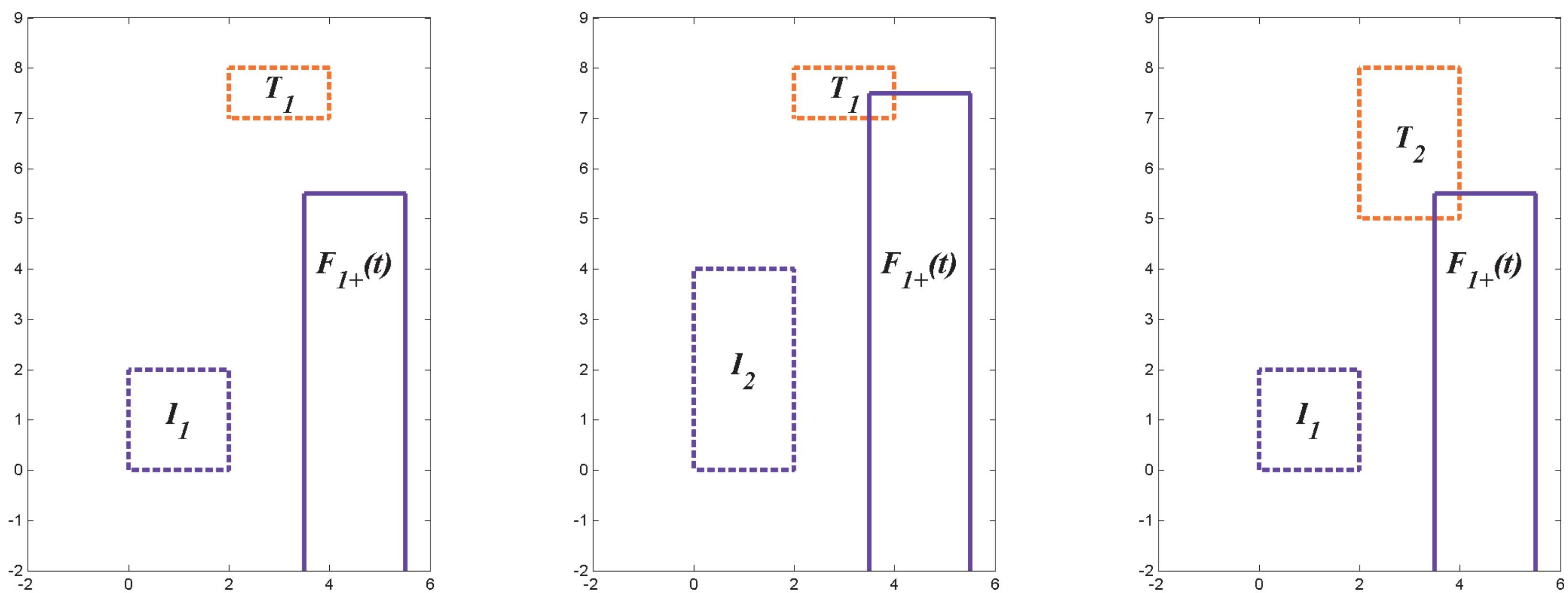
$$I_1 = [0, +2] \times [0, +2]$$

$$I_2 = [0, +2] \times [0, +4]$$

$$T_1 = [+2, +4] \times [+7, +8]$$

$$T_2 = [+2, +4] \times [+5, +8]$$

Forward Reach Set Results



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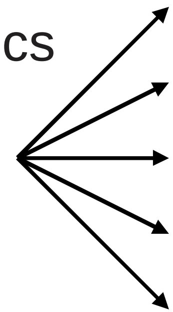

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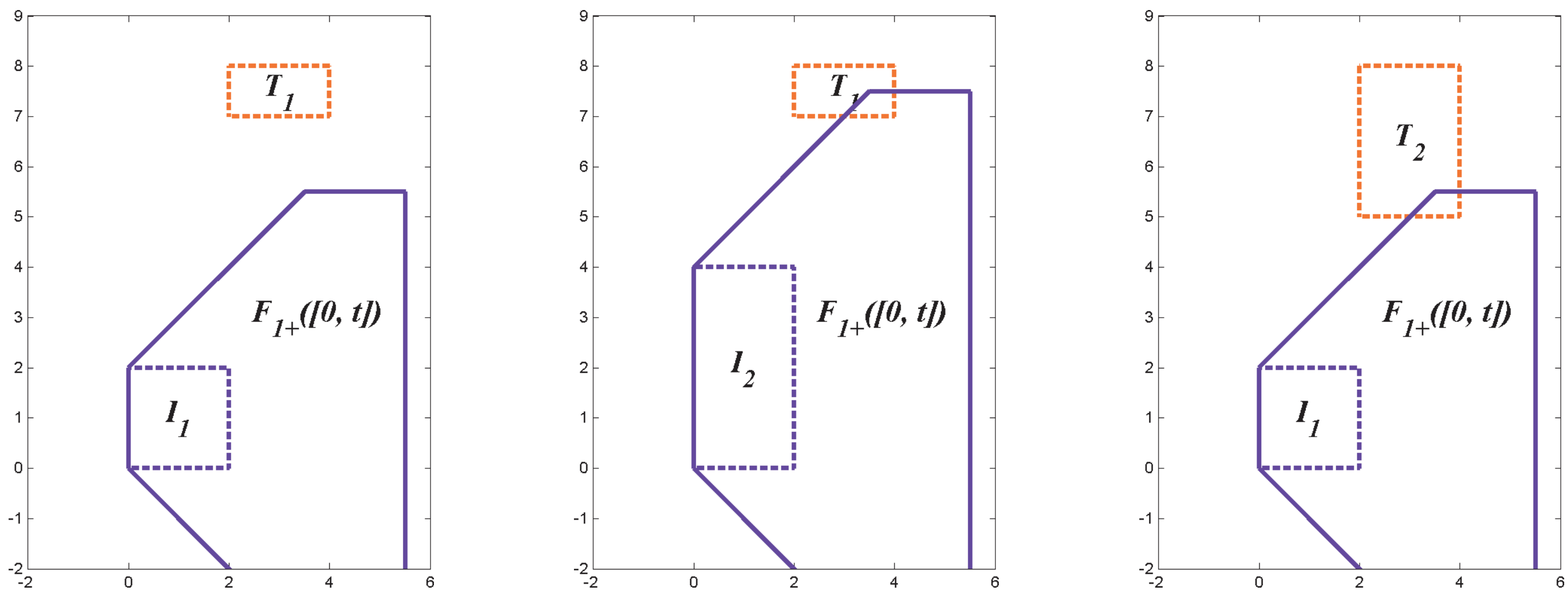
Forward Reach Tube Results



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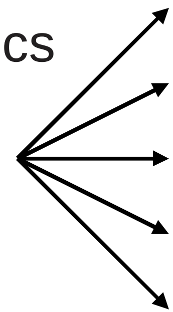

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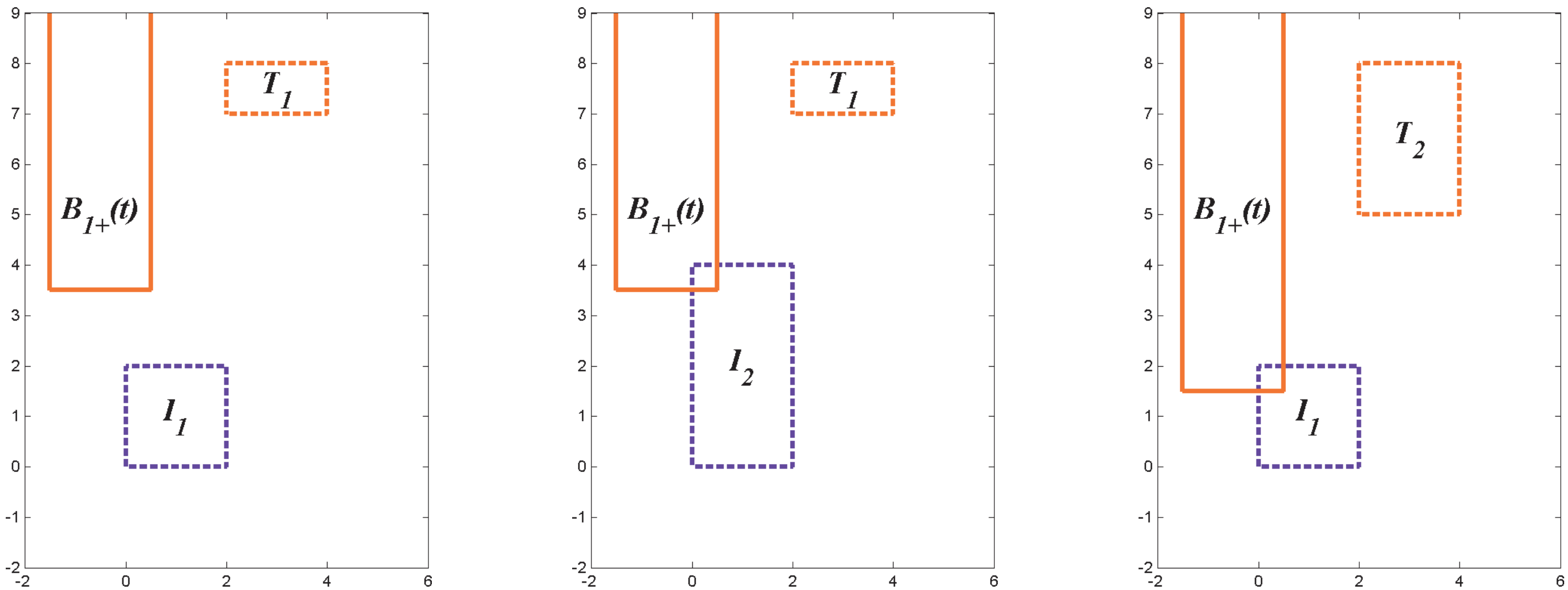

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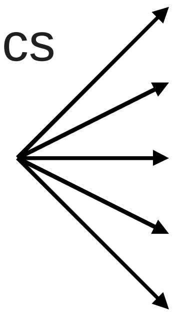

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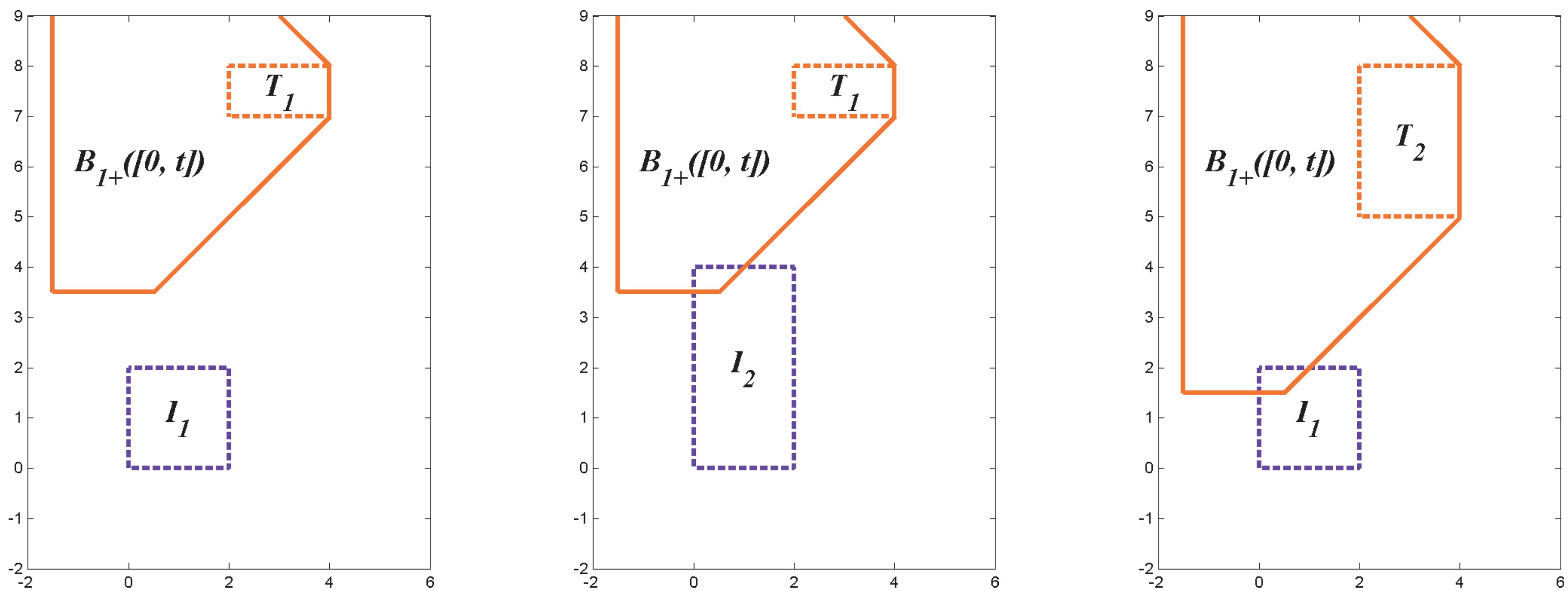
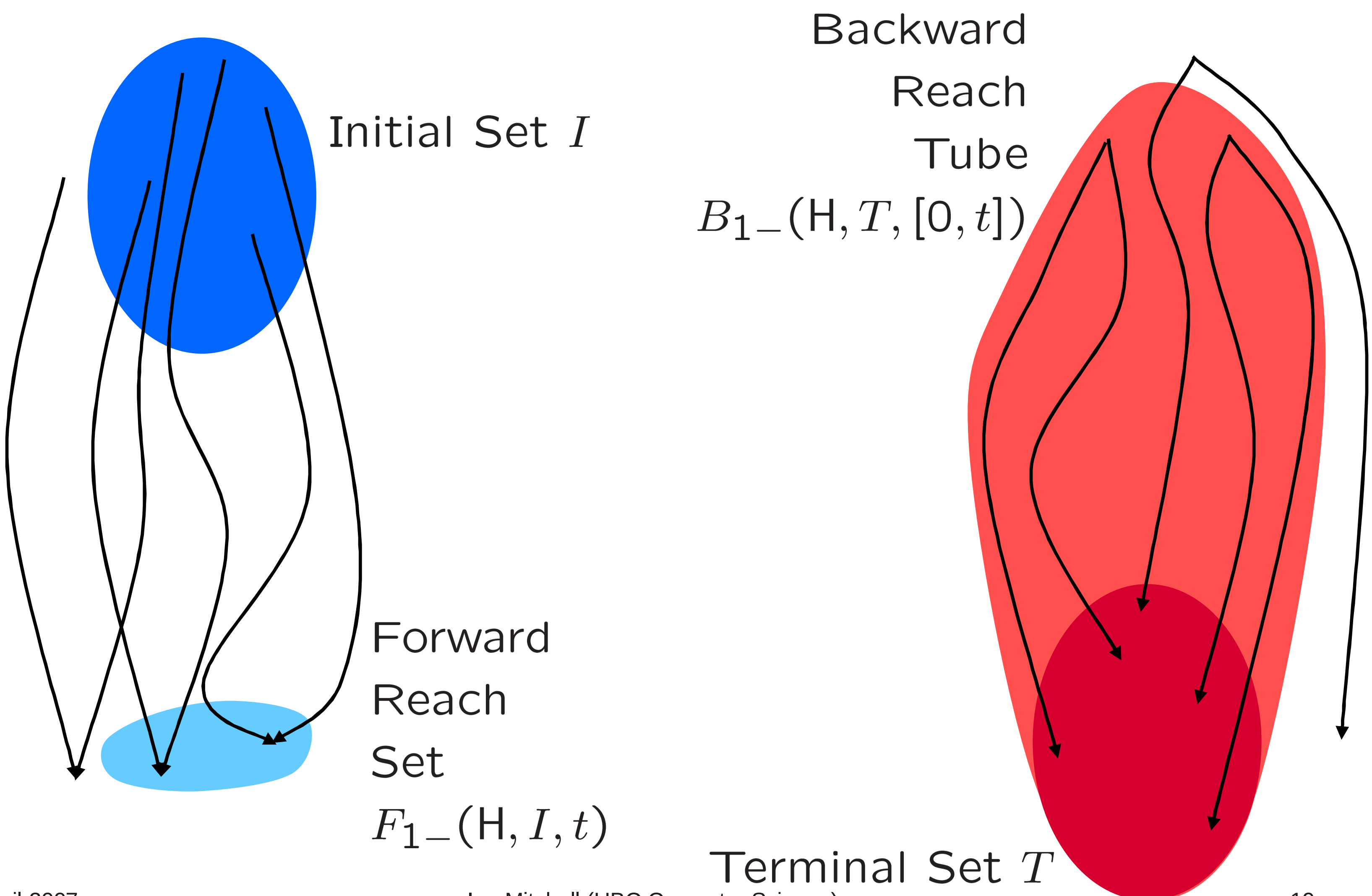
Backward Reach Tube Results



Minimal Reachability

- Input signal $u(\cdot)$ minimizes size of the set or tube



Minimal Reachability Definition

$$F_{1-}(\mathsf{H}, S, t) \triangleq \{\hat{z} \in \mathbb{Z} \mid \forall u(\cdot) \in \mathbb{U}, \exists z \in S, \\ \xi_{\mathsf{H}}(t; z, 0, u(\cdot)) = \hat{z}\}$$

$$F_{1-}(\mathsf{H}, S, [0, t]) \triangleq \{\hat{z} \in \mathbb{Z} \mid \forall u(\cdot) \in \mathbb{U}, \exists z \in S, \exists s \in [0, t], \\ \xi_{\mathsf{H}}(s; z, 0, u(\cdot)) = \hat{z}\}$$

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Minimal results also apply to systems with adversarial inputs; for example:

$$B_{2}(\mathsf{H}, S, [0, t]) \triangleq \{z \in \mathbb{Z} \mid \exists v(\cdot) \in \mathbb{V}, \forall u(\cdot) \in \mathbb{U}, \\ \exists \hat{z} \in S, \exists s \in [0, t], \\ \xi_{\mathsf{H}}(0; z, -s, u(\cdot), v(\cdot)) = \hat{z}\}.$$

Minimal Reachability Results

- Reach tubes provide more information

$$\bigcup_{\hat{t} \in [0,t]} F_{1-}(\mathsf{H}, S, \hat{t}) \subseteq F_{1-}(\mathsf{H}, S, [0,t])$$

$$\bigcup_{\hat{t} \in [0,t]} B_{1-}(\mathsf{H}, S, \hat{t}) \subseteq B_{1-}(\mathsf{H}, S, [0,t])$$

  - Choice of trajectory length t is quantified first for sets but last for tubes

Minimal Reachability Results

- Backward reach tubes are the only minimal reachability operator that can prove that there exists an input $u(\cdot)$ which keeps the system safe

$$B_{1-}(\mathsf{H}, T, [0,t]) \cap I = \emptyset \iff \exists u(\cdot) \in \mathbb{U} \text{ that keeps H safe}$$

$$B_{1-}(\mathsf{H}, T, s) \cap I = \emptyset, \ \forall s \leq t \Longleftarrow \exists u(\cdot) \in \mathbb{U} \text{ that keeps H safe}$$

$$\left.\begin{array}{r} F_{1-}(\mathsf{H}, I, t) \\ F_{1-}(\mathsf{H}, I, [0,t]) \end{array}\right\} \text{ provide no relevant information}$$

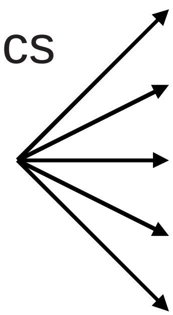
  – Basic problem with minimal forward reachability: the state lying in the terminal set is chosen before the input, while the state lying in the initial set is chosen after

Minimal Reachability Demonstration

System Dynamics

$$\dot{x} = \begin{bmatrix} +1 \\ u \end{bmatrix}$$

$$|u| \leq 1$$



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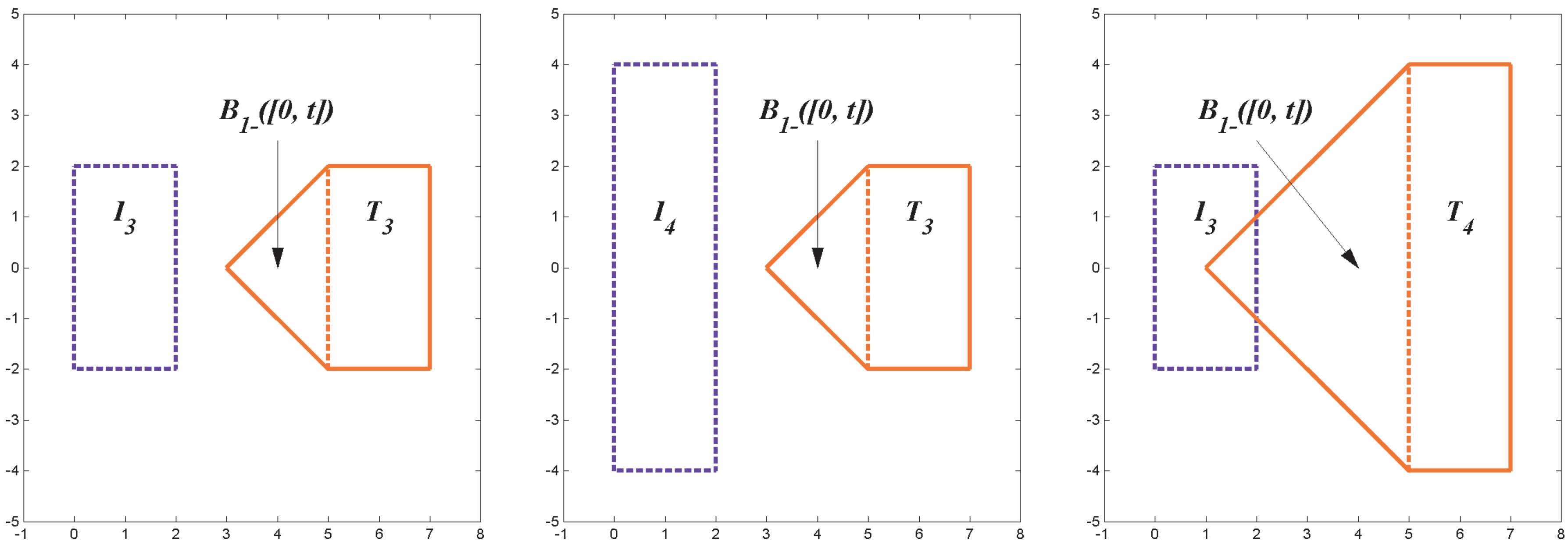
(Correct) Backward Reach Tube Results



Minimal Reachability Demonstration

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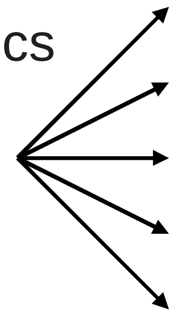

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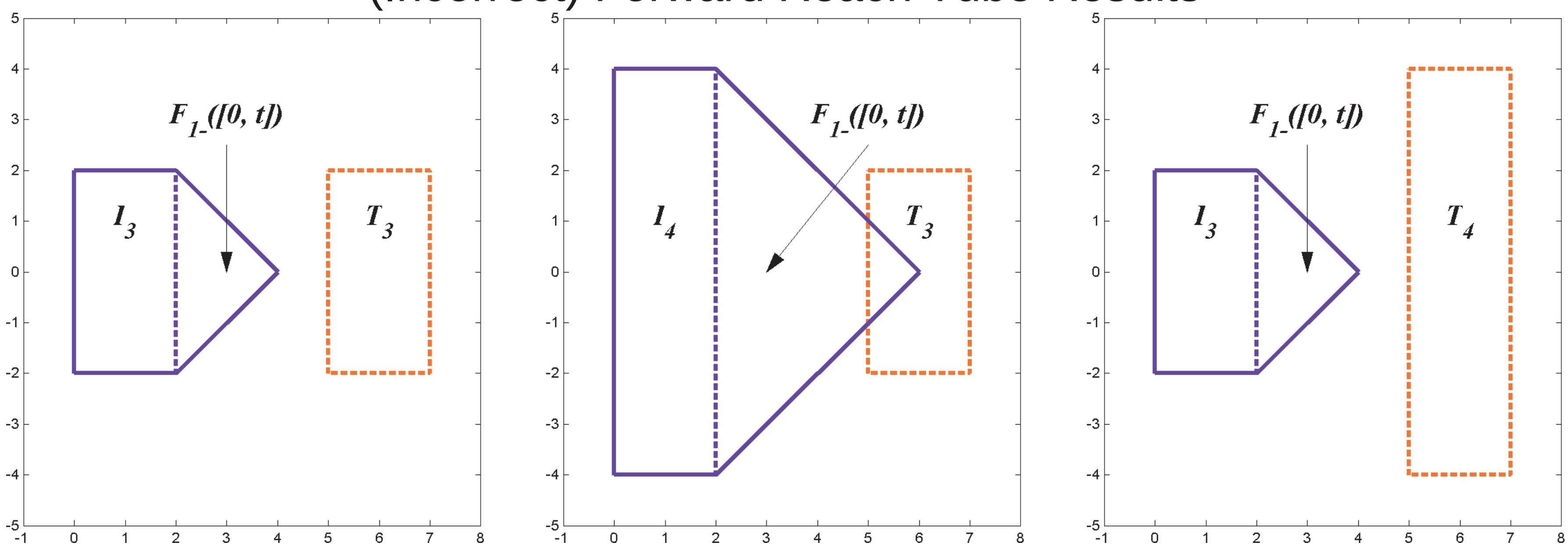
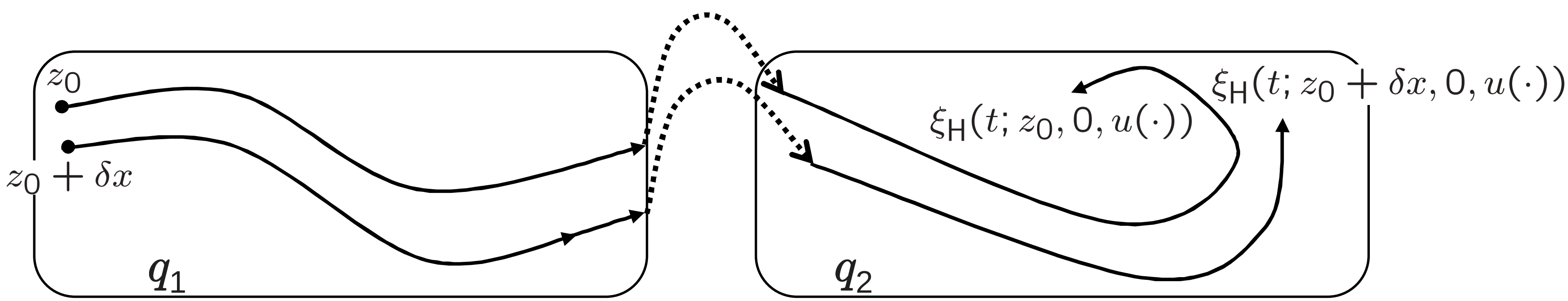
(Incorrect) Forward Reach Tube Results



Trajectory Sensitivity

- To approximate reach sets and tubes, direct algorithms integrate trajectories
- Small(?) perturbations occur in representation
  - Floating point roundoff
  - Simplified dynamics
  - Approximating the true set with a larger set from the appropriate class
- How might the interaction of perturbations and dynamics affect the quality of the approximation?

Compare $\xi_{\mathsf{H}}(t; z_0 + \delta x, 0, u(\cdot))$ to $\xi_{\mathsf{H}}(t; z_0, 0, u(\cdot))$



Sensitivity Analysis

- Focus on effect of continuous perturbation

$$\xi_H(t; z_0 + \delta x, 0, u(\cdot)) = \xi_H(t; z_0, 0, u(\cdot)) + \Xi_H(t; \xi_H(\cdot))\delta x + \mathcal{O}(\delta x^2)$$

- Sensitivity matrix

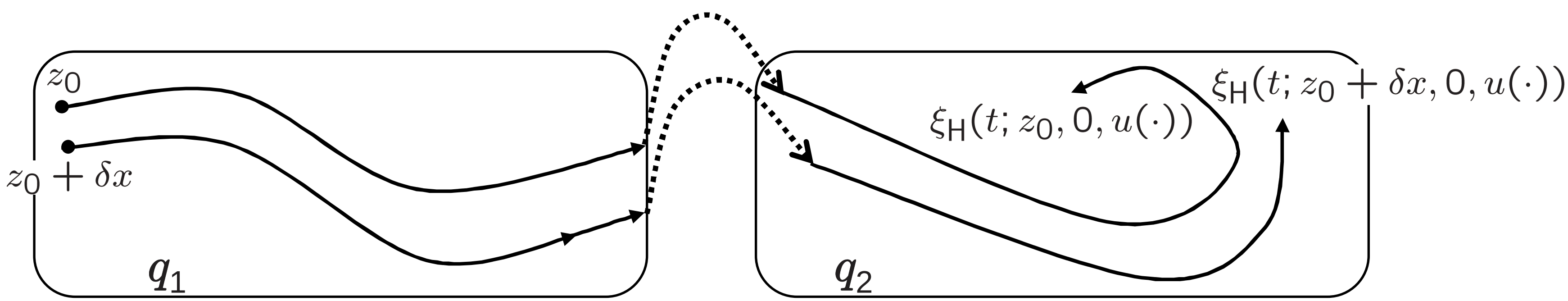
$$\Xi_H(t; \xi_H(\cdot)) \triangleq \frac{\partial \xi_H(t; z_0, 0, u(\cdot))}{\partial x_0}.$$

- Sensitivity of system dynamics

$$\mathbf{F}(q, x, u) \triangleq \mathbf{D}_x f(q, x, u) \qquad \mathbf{R}(q, \hat{q}, x, u) \triangleq \mathbf{D}_x r(q, \hat{q}, x, u)$$

- Continuous evolution of sensitivity matrix

$$\frac{d}{dt}\Xi_H(t) = \mathbf{F}(q, x, u)\Xi_H(t) \qquad \Xi_H(0) = \mathbf{I}$$



Sensitivity Analysis

- Discrete evolution of sensitivity matrix
  - Switching surfaces (guards, domains) specified implicitly

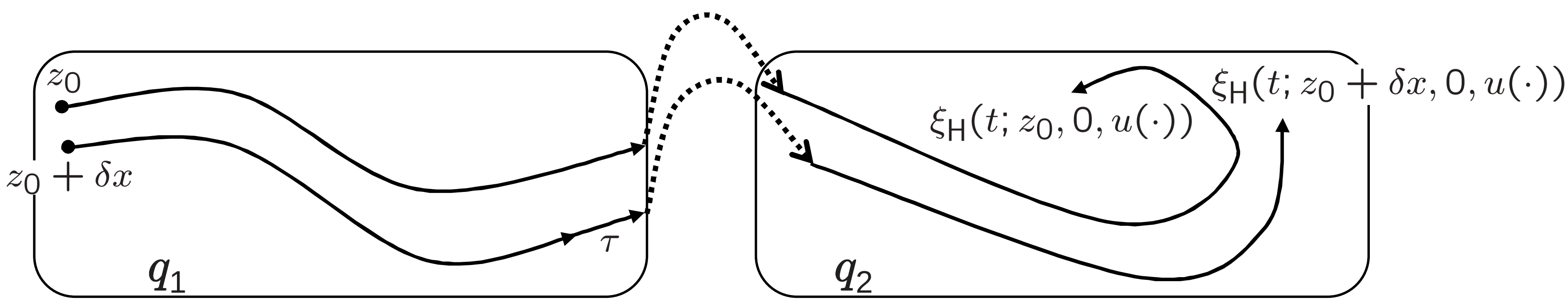
$$D(q, u_D) = \{x \in \mathbb{X} \mid \psi_D(q, x, u_D) \leq 0\}$$
$$G(q, \hat{q}, u_D) = \{x \in \mathbb{X} \mid \psi_G(q, \hat{q}, x, u_D) \leq 0\}$$

  - Difference in switching time [Hiskins & Pai, IEEE TC&S 2000]

$$\tau = \frac{\partial t(z_0)}{\partial z_0} = -\frac{\nabla \psi(x^-)^T \, \Xi_\mathsf{H}(t^-)}{\nabla \psi(x^-)^T f(q^-, x^-, u)}$$

  - Jump in sensitivity

$$\Xi_\mathsf{H}(t^+) = \mathbf{R}(q^-, q^+, x^-, u) \left(\Xi_\mathsf{H}(t^-) + f(q^-, x^-, u)\tau\right) - f(q^+, x^+, u)\tau,$$



Sensitivity of Forward Reachability

$$\|\xi_{\mathsf{H}}(s; z+\delta x, t, u(\cdot)) - \xi_{\mathsf{H}}(s; z, t, u(\cdot))\| \leq \|\Xi_{\mathsf{H}}(s; \xi_{\mathsf{H}}(\cdot))\| \|\delta x\|.$$

- Sensitivity matrix can become large via

$$\begin{aligned} \text{Real}[\lambda(\mathbf{F})] \gg 0 &\quad \text{continuous evolution} \\ |\lambda(\mathbf{R})| \gg 1 &\quad \text{discrete jumps} \\ \nabla\psi^T f^- \approx 0 &\quad \text{grazing contact with switching surface} \end{aligned}$$

- Systems satisfying these properties are inherently unpredictable
  - Deterministic models are rarely used for such systems

Sensitivity of Backward Reachability

- System dynamics are reversed

$$\overleftarrow{f} = -f \implies \overleftarrow{\mathbf{F}} = -\mathbf{F} \implies \lambda(\overleftarrow{\mathbf{F}}) = -\lambda(\mathbf{F})$$
$$\mathbf{R}\overleftarrow{\mathbf{R}} = \mathbf{I} \implies \overleftarrow{\mathbf{R}} = \mathbf{R}^{-1} \implies \lambda(\overleftarrow{\mathbf{R}}) = \lambda(\mathbf{R})^{-1}$$

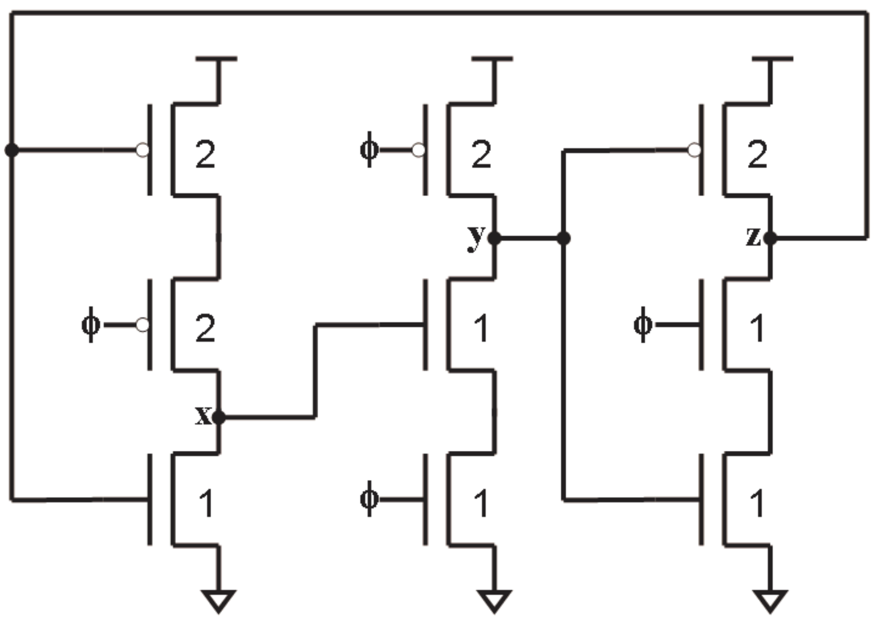
- Sensitivity matrix can become large via

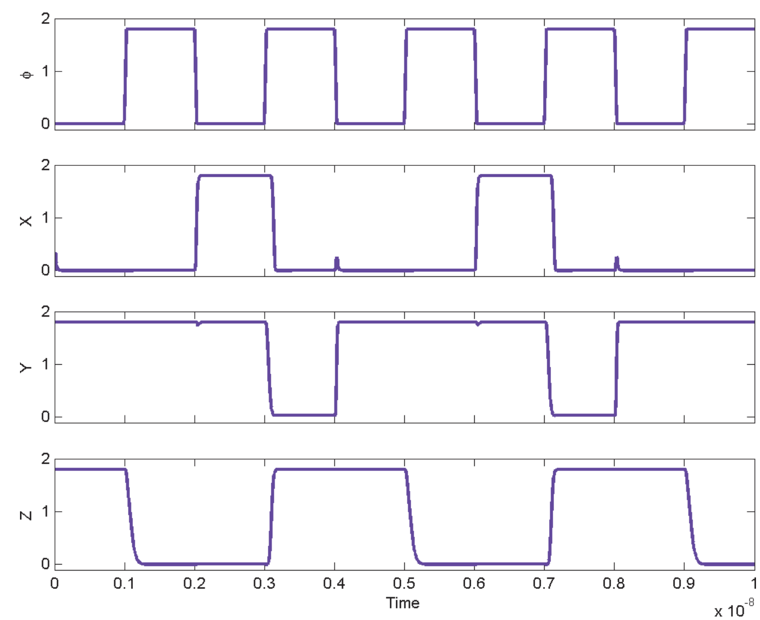
$$\begin{aligned} \text{Real}[\lambda(\mathbf{F})] \ll 0 &\quad \text{backward continuous evolution} \\ |\lambda(\mathbf{R})| \ll 1 &\quad \text{backward discrete jumps} \\ \nabla\psi^T f^+ \approx 0 &\quad \text{backward grazing contact} \\ &\quad \text{with switching surface} \end{aligned}$$

- Systems which show contraction are likely to be ill-conditioned for backward reachability
  - Such systems are commonly encountered, because their models are well-conditioned in forward time

Continuous System Sensitivity Example

- Toggle circuit [Yuan & Svensson, IEEE JSSC 1998]
  - Period of output **z** is double period of input ϕ
  - Short channel transistor model with velocity saturation, all capacitance to ground and interconnect capacitance is ignored [Hodges, Jackson & Saleh, 3rd edition 2004]
  - Forward verification that chain of toggles can operate as a counter [Greenstreet, CAV 1996]
  - Thanks: Mark Greenstreet, Chao Yan & Suwen Yang for simulation

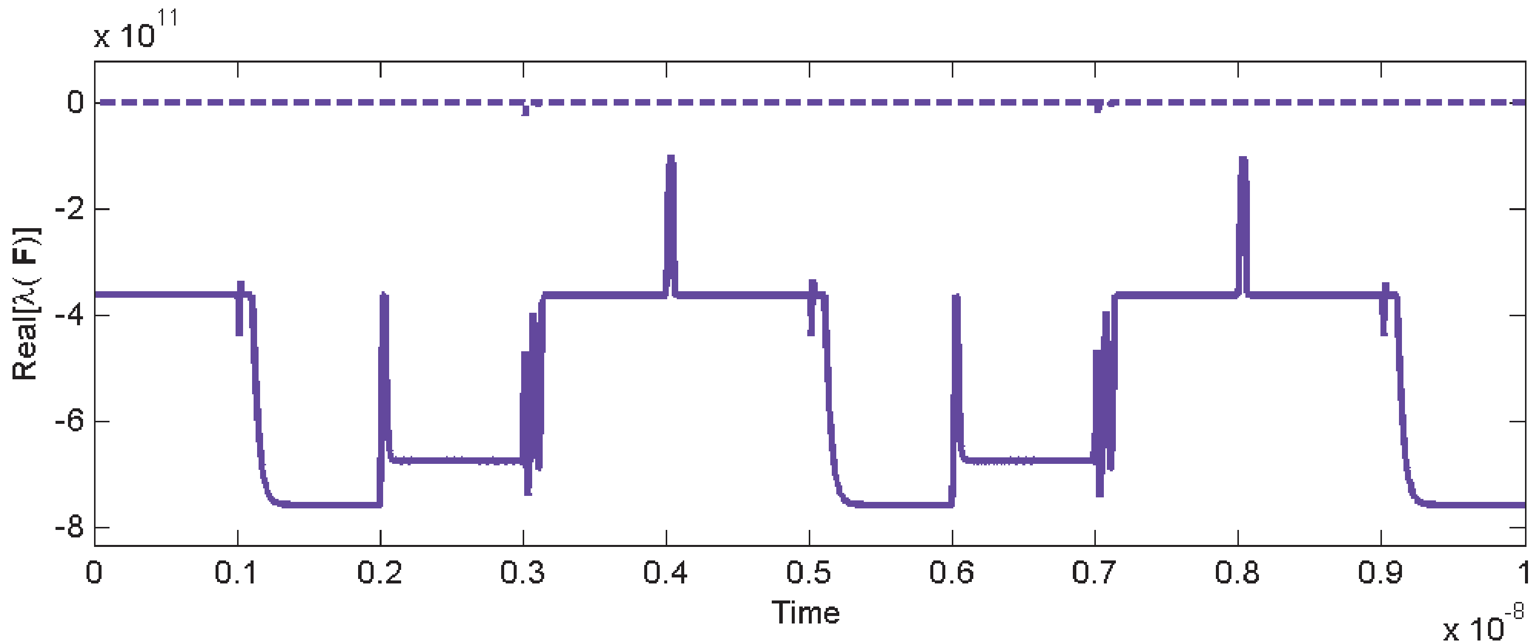




Toggle Circuit Sensitivity

- System dynamics has components which are strongly contractive
  - Sensitivity matrix of continuous dynamics has eigenvalues with large negative real component

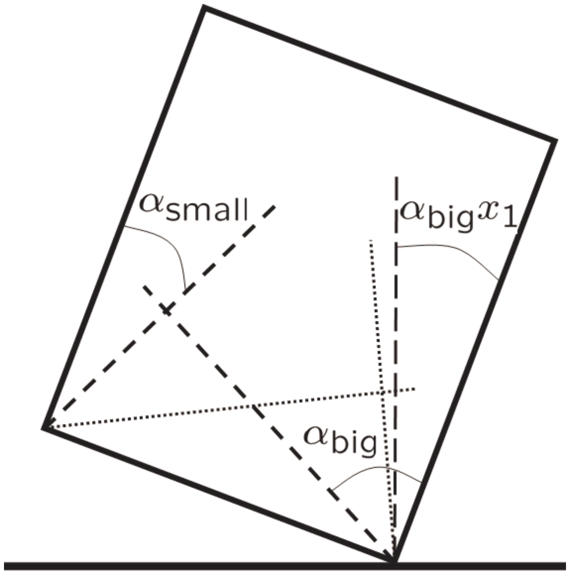
$$\mathbf{F}(q, x, u) \triangleq \mathbf{D}_x f(q, x, u)$$

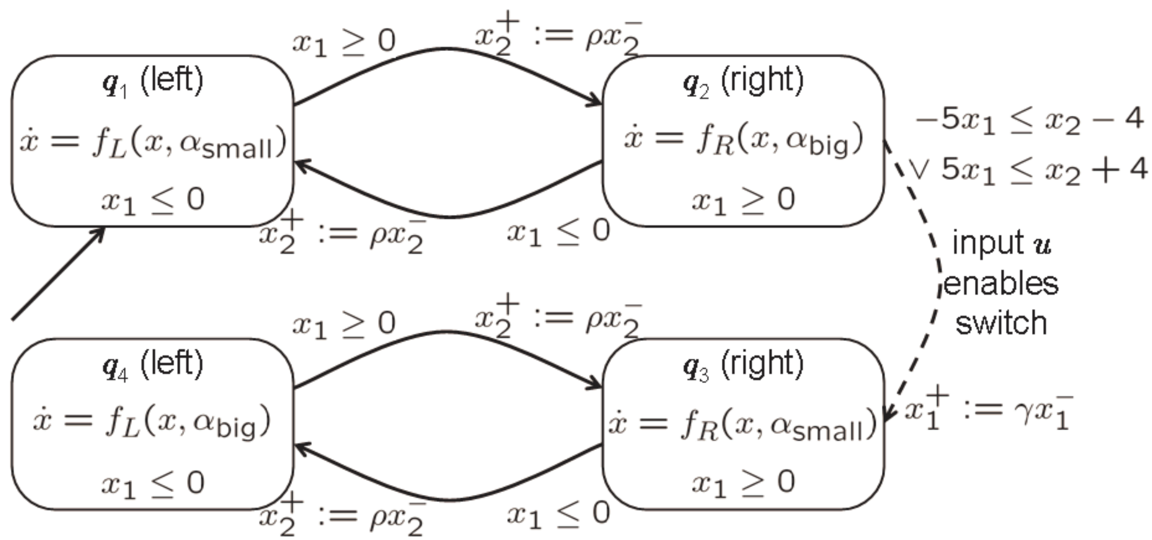


- Backward reachability will be ill-conditioned

Discrete System Sensitivity Example

- Adapted from rocking block in [Lygeros, Johansson, Simic, Zhang & Sastry, IEEE TAC 2003]
  - Discrete control input can change location of center of mass



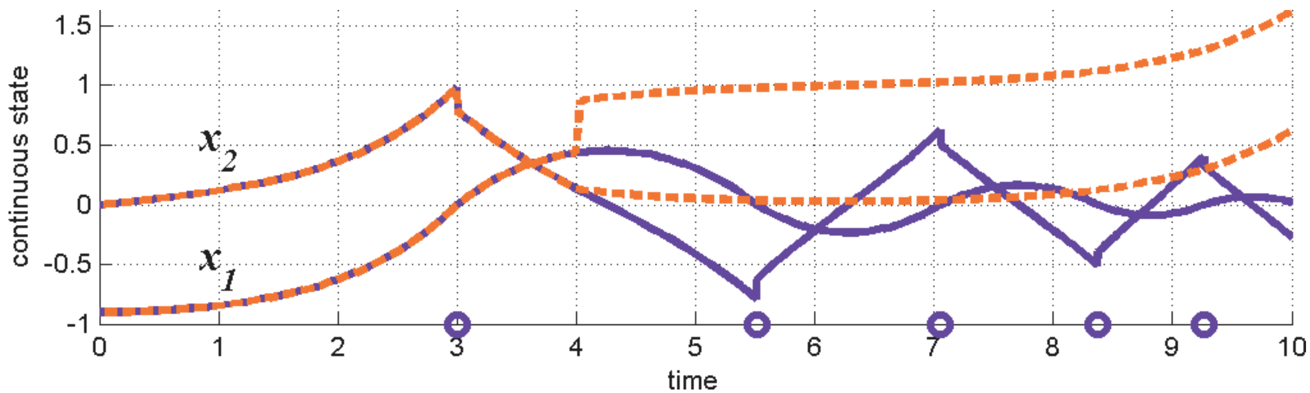


$$f_L(x, \alpha) = \begin{bmatrix} x_2 \\ \frac{1}{\alpha} \sin(\alpha(1 + x_1)) \end{bmatrix} \qquad f_R(x, \alpha) = \begin{bmatrix} x_2 \\ -\frac{1}{\alpha} \sin(\alpha(1 - x_1)) \end{bmatrix}$$

$$\alpha_{\text{big}} = \pi/3, \quad \alpha_{\text{small}} = \pi/6, \quad \rho = 0.8, \quad \gamma = \alpha_{\text{big}}/\alpha_{\text{small}} = 2.$$

Rocking Block Sensitivity

- Two typical trajectories
  - Constant center of mass (blue) or switched (red)



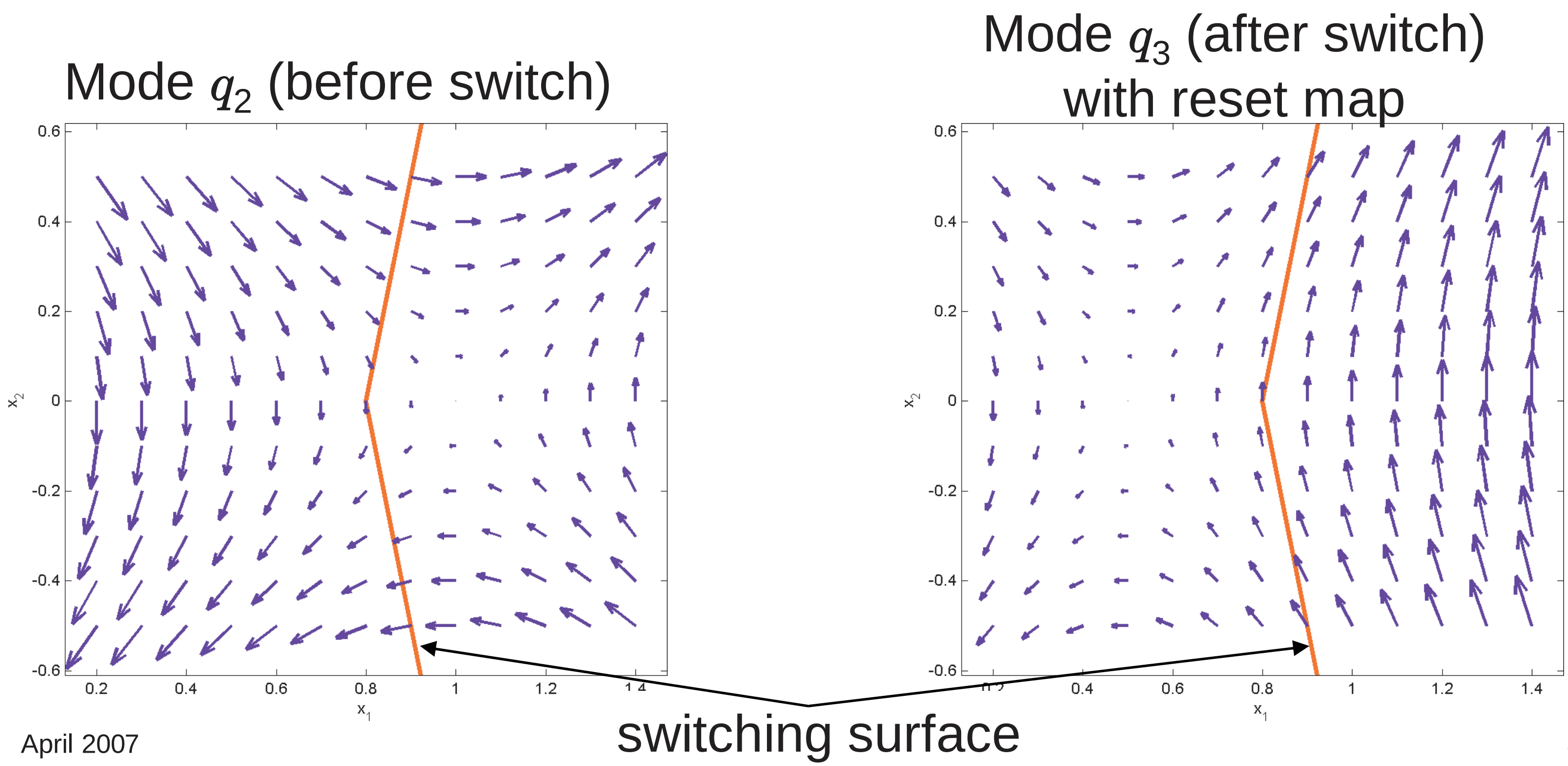
- Forward behaviour
  - Final state is sensitive to initial conditions (tipped or not)
  - Switching (controlled or autonomous) is not locally sensitive
- Backward behaviour
  - Controlled switch is sensitive through interaction with reset
  - Reset is sensitive for $\rho \ll 1$

Switching Surface Sensitivity

- Backward switching sensitivity is not obvious

$$\overleftarrow{\tau} = \frac{\partial t(z_0)}{\partial z_0} = \frac{\nabla \psi(x^+)^T \, \Xi_{\mathsf{H}}(t^+)}{\nabla \psi(x^+)^T f(q^+, x^+, u)}$$

$$\nabla \psi(x^+)^T f(q^+, x^+, u) = 0 \text{ at } x^+ \approx \frac{1}{23}\begin{bmatrix} 19 \\ 3 \end{bmatrix}$$



Conclusions

- All reachability operators are effective for proving universal safety over all input signals
- Only backward reach tube is effective for proving existence of a safe input signal
- For typical models, ill conditioning is more likely to occur for backward operators
- Results depend on the desired operator, not the algorithm

Comparing Forward and Backward Reachability as Tools for Safety Analysis

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