Optimal Stopping

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Overview

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WHY?

- **House selling**: You have a house and wish to sell it. Each day you are offered X_n for your house, and pay k to continue advertising it. If you sell your house on day n, you will earn y_n, where $y_n = (X_n - nk)$. You wish to maximise the amount you earn by choosing a stopping rule.
- **Secretary Problem**: You are observing a sequence of objects which can be ranked from best to worst. You wish to choose a stopping rule which maximises your chance of picking the best object.

WHAT? (Problem Definition)

- Stopping Rule is defined by two objects
  - A sequence of random variables, X_1, X_2, .., whose joint distribution is assumed known
  - A sequence of real-valued reward functions,

$$y_0,\ y_1(x_1),\ y_2(x_1,x_2),\ \ldots,\ y_\infty(x_1,x_2,\ldots)$$

- Given these objects, the problem is as follows:
  - You are observing the sequence of random variables, and at each step n, you can choose to either stop observing or continue
  - If you stop observing, you will receive the reward y_n
  - You want to choose a stopping rule Φ to maximize your expected reward (or minimize the expected loss)

Stopping Rule

- Stopping rule Φ consists of sequence

$$\Phi = (\Phi_0, \Phi_1(x_1), \Phi_2(x_1,x_2), \ldots)$$

  - $\Phi_n(x_1,\ldots,x_n)$: probability you stop after step n
  - $0 <= \Phi_n(x_1,\ldots,x_n) <= 1$
- If N is random variable over n (time to stop), the probability mass function ψ is defined as

$$\psi = (\psi_0, \psi_1, \psi_2, \ldots, \psi_\infty)$$

$$\psi_n(x_1,\ldots,x_n) = P(N = n | X_1 = x_1,\ldots, X_n = x_n)$$

$$\Rightarrow \psi_n(x_1,\ldots,x_n) = [\prod_1^{n-1}(1-\phi_j(x_1,\ldots,x_j))]\phi_n(x_1,\ldots,x_n)$$

HOW?
(Solution Framework)

- So, the problem is to choose stopping rule Φ to maximize the expected return, V(Φ), defined as

$$V(\phi) = E\sum_{j=0}^{\infty}\{\psi_j(X_1,\dots,X_j) * y_j(X_1,\dots,X_j)\}$$

j = ∞ if we never stop (infinite horizon)
j = T if we stop at T (finite horizon)

Optimal Stopping in Finite Horizon

- Special case of general problem, by setting $y_{T+1} = y_{T+2} = \ldots\ y_{\infty} = -\infty$
- Use DP algorithm

$$V_T = y_T(x_1,\ldots,x_T)$$

$$V_j(x_1,\ldots,x_j) = \max\{y_j(x_1,\ldots,x_j), E(V_{j+1}(x_1,\ldots,x_{j+1}))\}$$

- Here, $V_j(x_1,\ldots,x_j)$ represents the maximum return one can obtain starting from stage j having observed $X_1=x_1,\ldots, X_j=x_j$.

The Classical Secretary Problem (CSP)

- Aka marriage problem, hiring problem, the sultan's dowry problem, the fussy suitor problem, and the best choice problem
- Rules of the game:
  - There is a single secretarial position to fill.
  - There are n applicants for the position, n is known.
  - The applicants can be ranked from best to worst with no ties.
  - The applicants are interviewed sequentially in a random order, with each order being equally likely.
  - After each interview, the applicant is accepted or rejected.
  - The decision to accept or reject an applicant can be based only on the relative ranks of the applicants interviewed so far.
  - Rejected applicants cannot be recalled.
  - The object is to select the best applicant. Win: If you select the best applicant. Lose: otherwise
- Note: An applicant should be accepted only if it is relatively best among those already observed

CSP – Solution Framework

- A relatively best applicant is called a **candidate**
- Reward Function
  - $y_j(x_1,\ldots,x_n) = j/n$ if applicant j is a candidate,
  - $= 0$ otherwise
- Lets say the interviewer rejects the first r-1 applicants and then accept the next relatively best applicant. We wish to find the optimal r

CSP – Solution Framework Cont.

Probability that the best applicant is selected is

$$P_r = \sum_{k=r}^{n} P(k^{th} \text{ applicant is best and selected})$$

$$= \sum_{k=r}^{n} P(k^{th} \text{ applicant is best})\, P(k^{th} \text{ applicant is selected} \mid \text{it is best})$$

$$= \sum_{k=r}^{n} \frac{1}{n} P(\text{best of first } k-1 \text{ appears before stage } r)$$

$$= \sum_{k=r}^{n} \frac{1}{\text{n}} \frac{r-1}{k-1} = \frac{r-1}{n} \sum_{k=r}^{n} \frac{1}{k-1}$$

** (r-1)/(r-1) = 1 if r = 1

CSP – Solution Framework Cont.

- For optimal r,

$$P_{r+1} \le P_r$$

$$\Rightarrow \frac{\text{r}}{\text{n}}\sum_{r+1}^{n}\frac{1}{k-1} \le \frac{r-1}{n}\sum_{r}^{n}\frac{1}{k-1}$$

$$\Rightarrow \sum_{r+1}^{n}\frac{1}{k-1} \le 1$$

$$r^* = \min\{r \ge 1 : \sum_{r+1}^{n}\frac{1}{k-1} \le 1\}$$

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
*r**	1	1	2	2	3	3	3	4	4
P	1.000	0.500	0.500	0.458	0.433	0.428	0.414	0.410	0.406

CSP – for large n

If n is large,

$$\sum_{r+1}^{n}\frac{1}{k-1}\approx \log(\frac{n}{r})$$

$$\Rightarrow \log(n/r^*) = 1$$

$$\Rightarrow r^* = n/e$$

$$P_{r^*} = e^{-1} = 0.368$$

References

- Optimal Stopping and Applications by Thomas S. Ferguson (http://www.math.ucla.edu/~tom/Stopping/Contents.html)
- http://en.wikipedia.org/wiki/Optimal_stopping
- http://en.wikipedia.org/wiki/Secretary_problem