Eulerian Approaches to Backward Reachability

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A Digression:
Approaches to Validating Designs

- By construction
  - property is inherent.
- By verification
  - property is provable.
- By simulation
  - check behavior for all inputs.
- By intuition
  - property is true. I just know it is.
- By assertion
  - property is true. Wanna make something of it?
- By intimidation
  - Don't even try to doubt whether it is true

Meret Oppenheim, *Object*, 1936

It is generally better to be higher in this list

Continuous Reach Sets and the Hamilton-Jacobi Equation

An Eulerian

Dynamic Implicit Surface

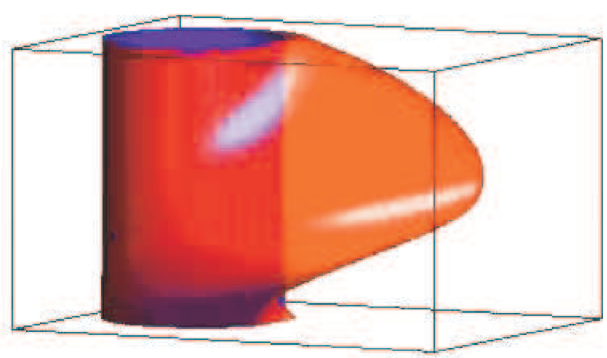
Framework

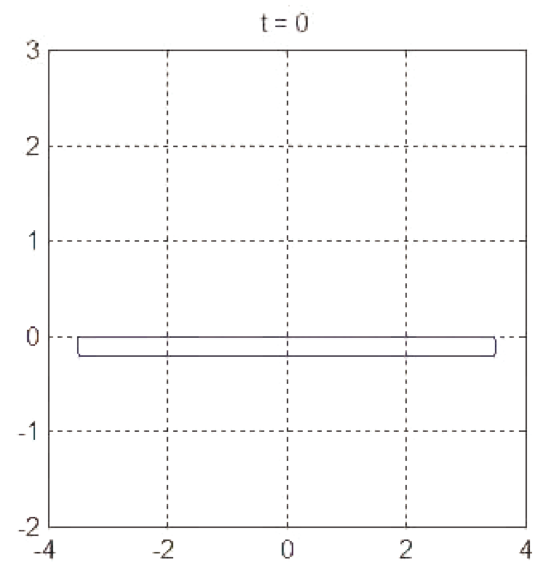
Outline

- Representation: Implicit Surface Functions
- Example
  - The game of two identical vehicles
- Evolution: the Time Dependent Hamilton-Jacobi Equation
  - Viscosity solutions and numerical methods
  - Modification for optimal stopping time
  - Alternative Eulerian schemes
- Applications of Reachability Analysis
  - Softwalls
  - ATC alerts
- Reducing the dimensional cost: projections

Calculating Reach Sets

- Two primary challenges
  - How to represent set of reachable states
  - How to evolve set according to dynamics
- Discrete systems $x_{k+1} = \delta(x_k)$
  - Enumerate trajectories and states
  - Efficient representations: Binary Decision Diagrams
- Continuous systems $dx/dt = f(x)$?



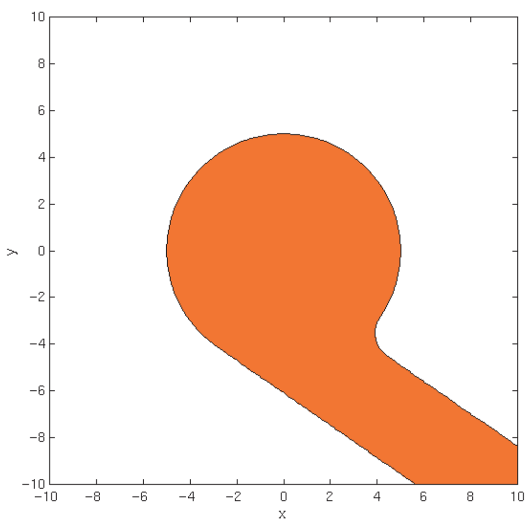


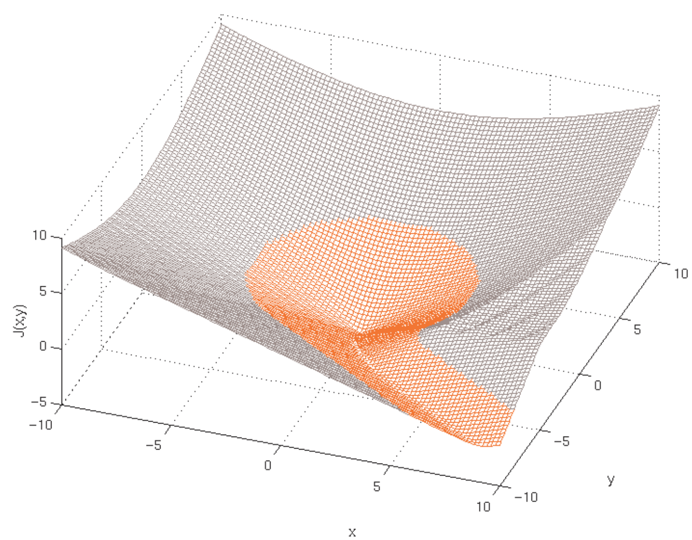
Implicit Surface Functions

- Set $G(t)$ is defined implicitly by an isosurface of a scalar function $\phi(x,t)$, with several benefits
  - State space dimension does not matter conceptually
  - Surfaces automatically merge and/or separate
  - Geometric quantities are easy to calculate

$$\phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$$

$$\mathcal{G}(t) = \{x \in \mathbb{R}^n \mid \phi(x,t) \leq 0\}$$



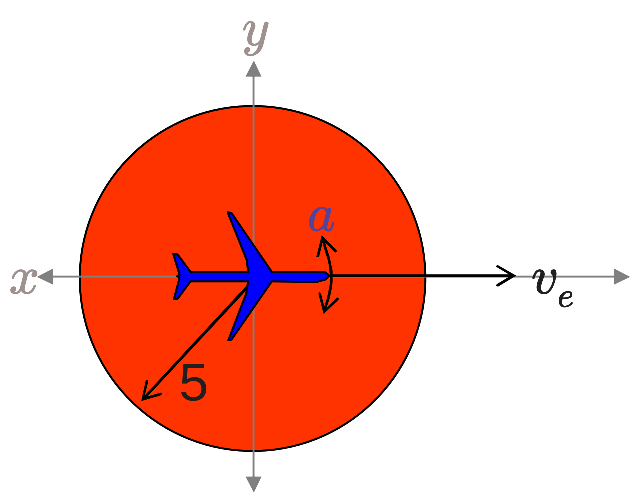


Game of Two Identical Vehicles

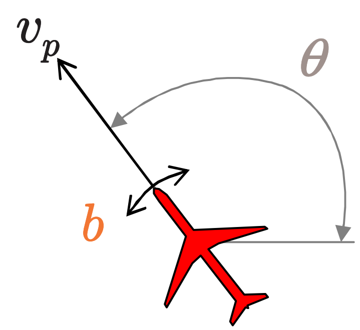
- Classical collision avoidance example
  - Collision occurs if vehicles get within five units of one another
  - Evader chooses turn rate $|a| \leq 1$ to avoid collision
  - Pursuer chooses turn rate $|b| \leq 1$ to cause collision
  - Fixed equal velocity $v_e = v_p = 5$

dynamics (pursuer)

$$\frac{d}{dt}\begin{bmatrix} x_p \\ y_p \\ \theta_p \end{bmatrix} = \begin{bmatrix} v_p \cos\theta_p \\ v_p \sin\theta_p \\ b \end{bmatrix}$$



evader aircraft (control)

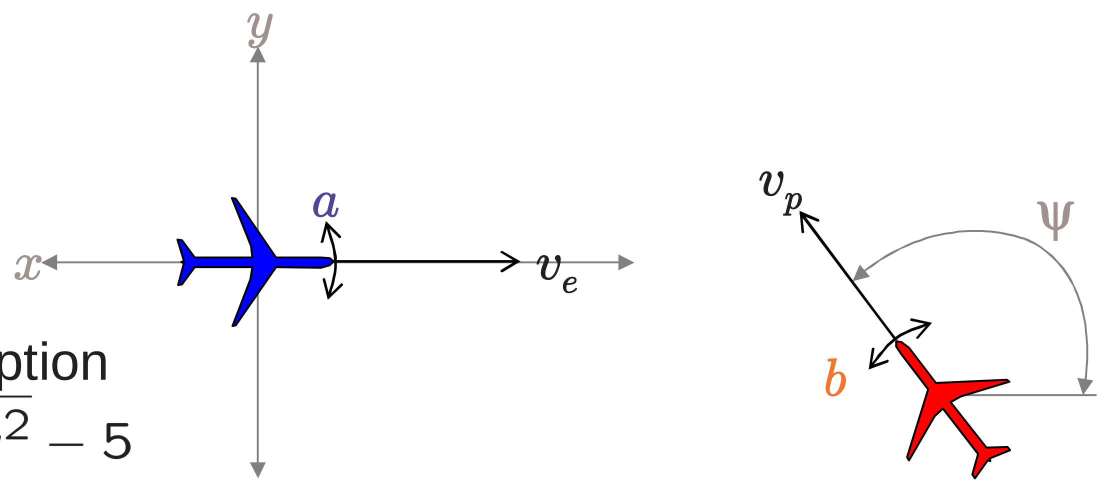


pursuer aircraft (disturbance)

Collision Avoidance Computation

- Work in relative coordinates with evader fixed at origin
  - State variables are now relative planar location (x,y) and relative heading ψ

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} -v_e + v_p \cos\psi - ay \\ v_p \sin\psi - ax \\ b - a \end{bmatrix}$$



target set description

$$h(x) = \sqrt{x^2 + y^2} - 5$$

evader aircraft (control)

pursuer aircraft (disturbance)

Evolving Reachable Sets

- Modified Hamilton-Jacobi partial differential equation

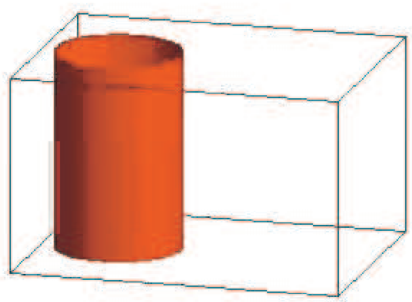
$$D_t\phi(x,t) + \min\left[0, H(x, D_x\phi(x,t))\right] = 0$$

$$\text{with Hamiltonian}: \; H(x,p) = \max_{a\in\mathcal{A}} \min_{b\in\mathcal{B}} f(x,a,b)\cdot p$$

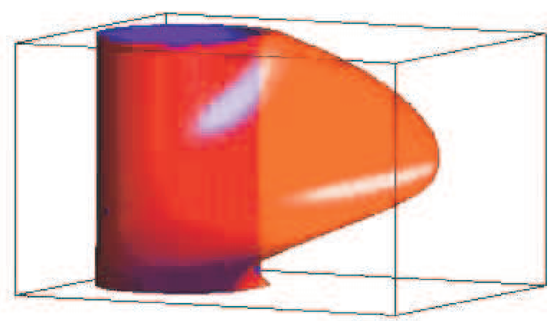
$$\text{and terminal conditions}: \; \phi(x,0) = h(x)$$

$$\text{where} \quad G(0) = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}$$

$$\text{and} \quad \dot{x} = f(x,a,b)$$



growth of reachable set



final reachable set

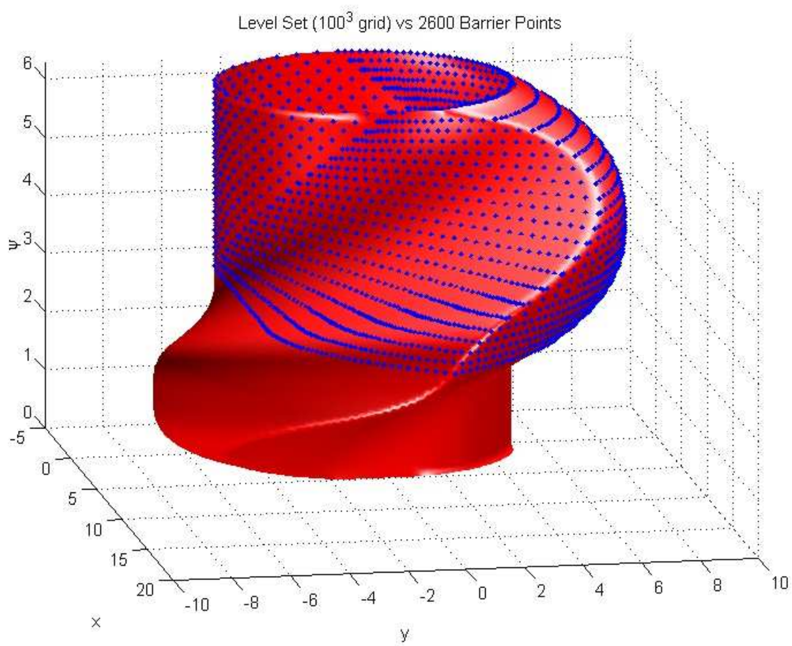
Time-Dependent Hamilton-Jacobi Eq'n

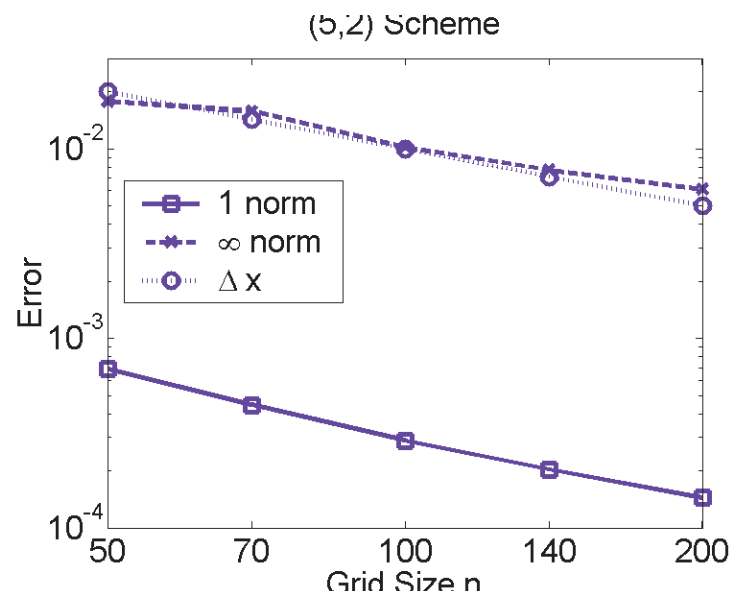
$$D_t\phi(x,t) + H(x, D_x\phi(x,t)) = 0$$

- First order hyperbolic PDE
  - Solution can form kinks (discontinuous derivatives)
  - For the backwards reachable set, find the "viscosity" solution [Crandall, Evans, Lions, …]
- Level set methods
  - Convergent numerical algorithms to compute the viscosity solution [Osher, Sethian, …]
  - Non-oscillatory, high accuracy spatial derivative approximation
  - Stable, consistent numerical Hamiltonian
  - Variation diminishing, high order, explicit time integration

Validating the Numerical Algorithm

- Analytic solution for reachable set can be found [Merz, 1972]
  - Applies only to identical pursuer and evader dynamics
  - Merz's solution placed pursuer at the origin, game is not symmetric
  - Analytic solution can be used to validate numerical solution
  - [Mitchell, 2001]





Solving a Differential Game

- Terminal cost differential game for trajectories $\xi_f(\cdot; x, t, a(\cdot), b(\cdot))$

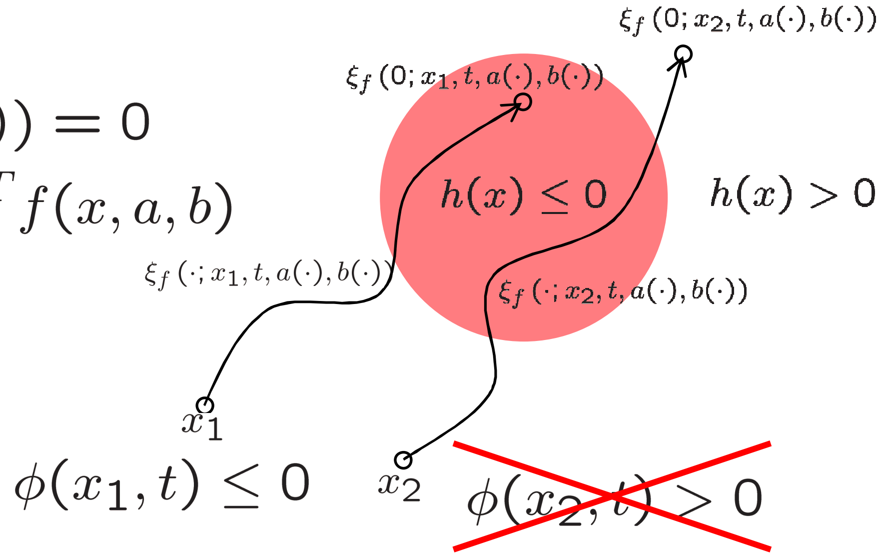
$$\phi(x,t) = \sup_{a(\cdot)} \inf_{b(\cdot)} h\left[\xi_f\left(0; x, t, a(\cdot), b(\cdot)\right)\right]$$

$$\text{where} \begin{cases} \xi_f\left(t; x, t, a(\cdot), b(\cdot)\right) = x \\ \dot{\xi}_f((s; x, t, a(\cdot), b(\cdot)) = f\left(x, a(s), b(s)\right) \\ \quad \text{terminal payoff function } h(x) \end{cases}$$

- Value function solution $\phi(x,t)$ given by viscosity solution to basic Hamilton-Jacobi equation
  - [Evans & Souganidis, 1984]

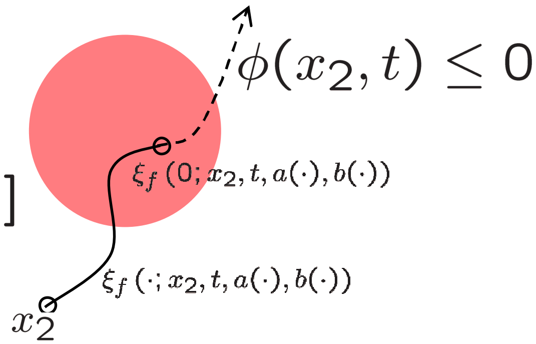
$$D_t\phi(x,t) + H\left(x, D_x\phi(x,t)\right) = 0$$

$$\text{where} \begin{cases} H(x,p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T f(x,a,b) \\ \phi(x,0) = h(x) \end{cases}$$



Modification for Optimal Stopping Time

- How to keep trajectories from passing through $G(0)$?
  - [Mitchell, Bayen & Tomlin IEEE TAC 2005]
  - Augment disturbance input



$$\tilde{b} = \begin{bmatrix} b & \underline{b} \end{bmatrix} \text{ where } \underline{b} : [t, 0] \to [0, 1]$$

$$\tilde{f}(x, a, \tilde{b}) = \underline{b} f(x, a, b)$$

  - Augmented Hamilton-Jacobi equation solves for reachable set

$$D_t\phi(x,t) + \tilde{H}(x, D_x\phi(x,t)) = 0 \text{ where } \begin{cases} \tilde{H}(x,p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T \tilde{f}(x, a, \tilde{b}) \\ \phi(x, 0) = h(x) \end{cases}$$

  - Augmented Hamiltonian is equivalent to modified Hamiltonian

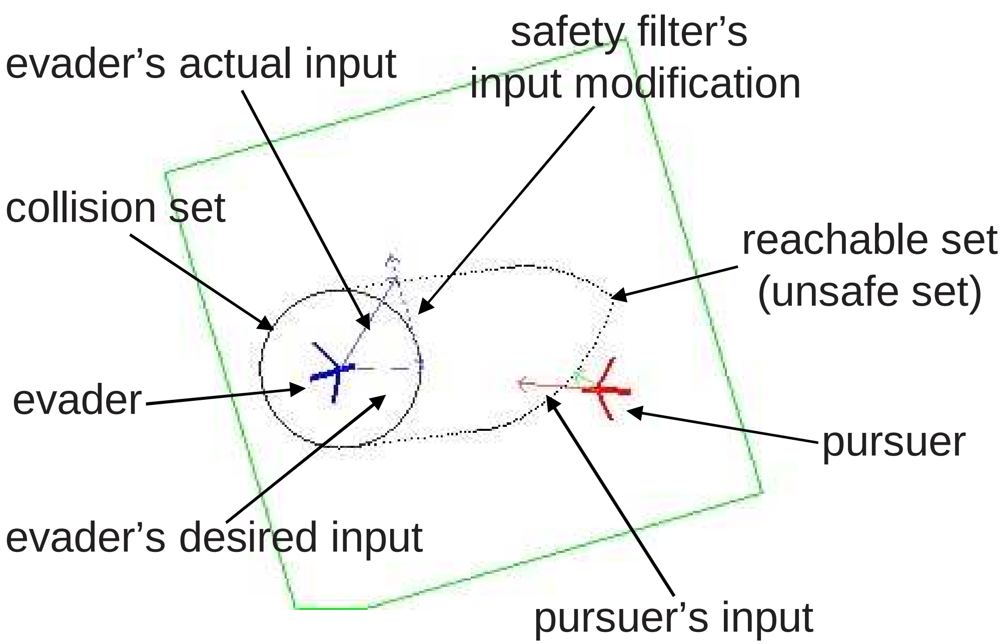
$$\begin{aligned} \tilde{H}(x,p) &= \max_{a \in \mathcal{A}} \min_{\tilde{b} \in \tilde{\mathcal{B}}} p^T \tilde{f}(x, a, \tilde{b}) \\ &= \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \min_{\underline{b} \in [0,1]} \underline{b} p^T f(x, a, b) \\ &= \min\left[0, \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T f(x, a, b)\right] = \min\left[0, H(x,p)\right] \end{aligned}$$

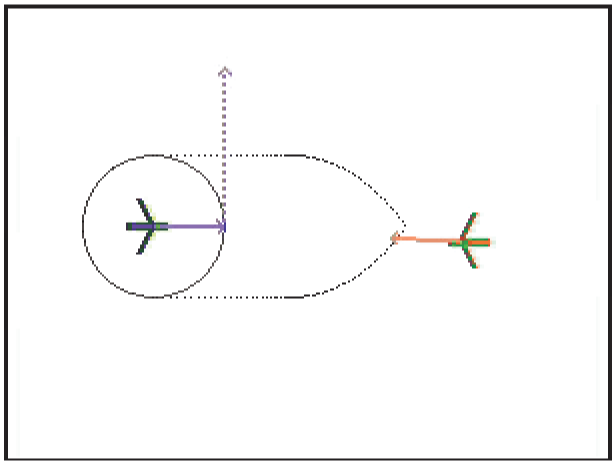
Alternative Eulerian Approaches

- Static Hamilton-Jacobi (Falcone, Sethian, …)
  - Minimum time to reach
  - (Dis)continuous implicit representation
  - Solution provides information on optimal input choices
- Viability kernels (Aubin, Saint-Pierre, …)
  - Based on set valued analysis for very general dynamics
  - Discrete implicit representation
  - Overapproximation guarantee
- Time-dependent Hamilton-Jacobi (this method)
  - Continuous solution
  - Information on optimal input choices available throughout entire state space
  - High order accurate approximations
- All three are theoretically equivalent

Application: Softwalls for Aircraft Safety

- Use reachable sets to guarantee safety
- Basic Rules
  - Pursuer: turn to head toward evader
  - Evader: turn to head east
- Evader's input is filtered to guarantee that pursuer does not enter the reachable set

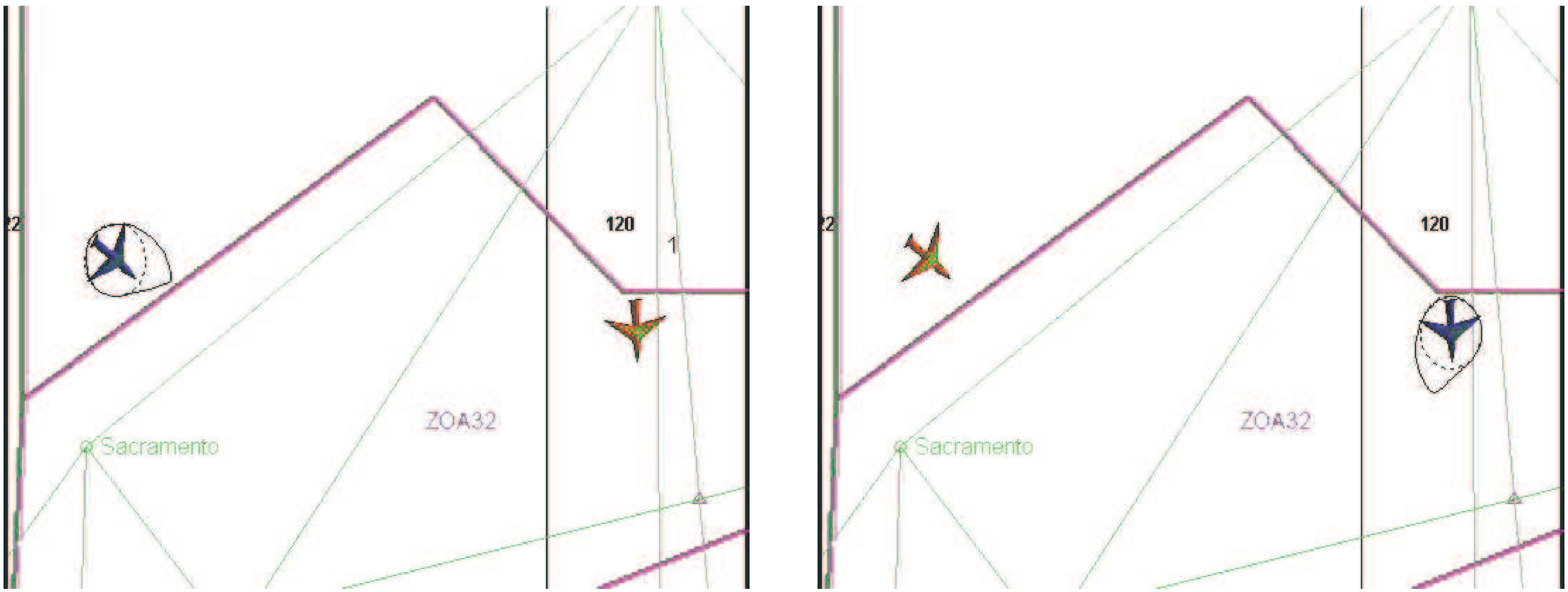




joint work with Edward Lee & Adam Cataldo

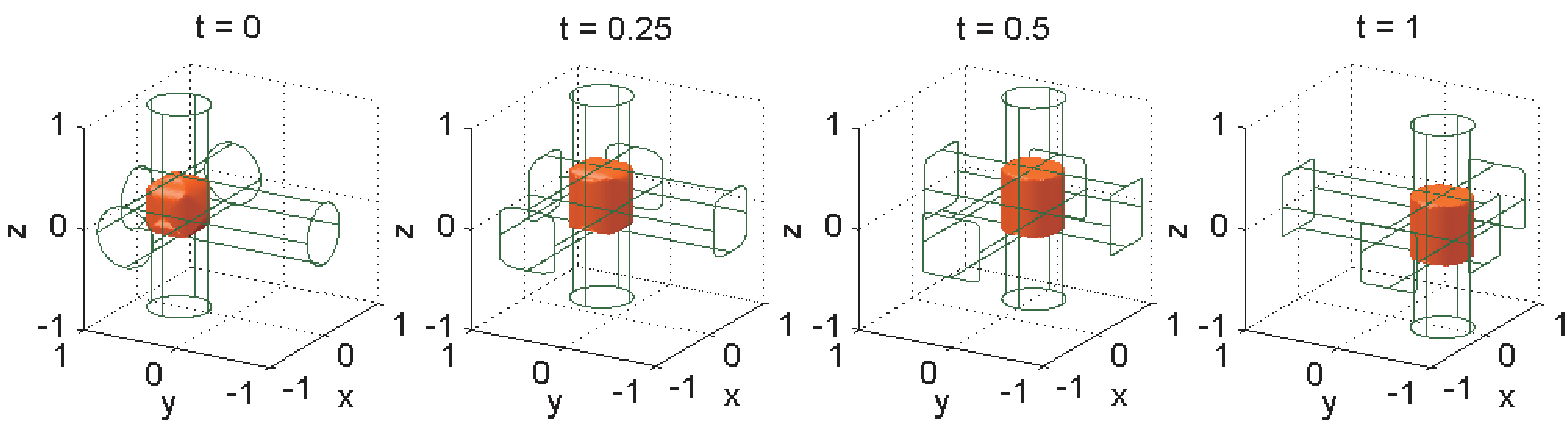
Application: Collision Alert for ATC

- Use reachable set to detect potential collisions and warn Air Traffic Control (ATC)
  - Find aircraft pairs in ETMS database whose flight plans intersect
  - Check whether either aircraft is in the other's collision region
  - If so, examine ETMS data to see if aircraft path is deviated
  - One hour sample in Oakland center's airspace—
    - 1590 pairs, 1555 no conflict, 25 detected conflicts, 2 false alerts



Projective Overapproximation

- Overapproximate reachable set of high dimensional system as the intersection of reachable sets for lower dimensional projections
  - [Mitchell & Tomlin, JSC 2003]
  - Example: rotation of “sphere” about z-axis



Hamilton-Jacobi in the Projection

- Consider x–z projection represented by level set $\phi_{xz}(x,z,t)$
  - Back projection into 3D yields a cylinder $\phi_{xz}(x,y,z,t)$
- Simple HJ PDE for this cylinder

$$D_t\phi_{xz}(x,y,z,t) + \sum_{i=1}^{3} p_i f_i(x,y,z) = 0 \quad \text{where} \begin{cases} p_1 = D_x\phi_{xz}(x,y,z,t) \\ p_2 = D_y\phi_{xz}(x,y,z,t) \\ p_z = D_z\phi_{xz}(x,y,z,t) \end{cases}$$

  - But for cylinder parallel to y-axis, $p_2 = 0$

$$D_t\phi_{xz}(x,y,z,t) + p_1 f_1(x,y,z) + p_3 f_3(x,y,z) = 0$$

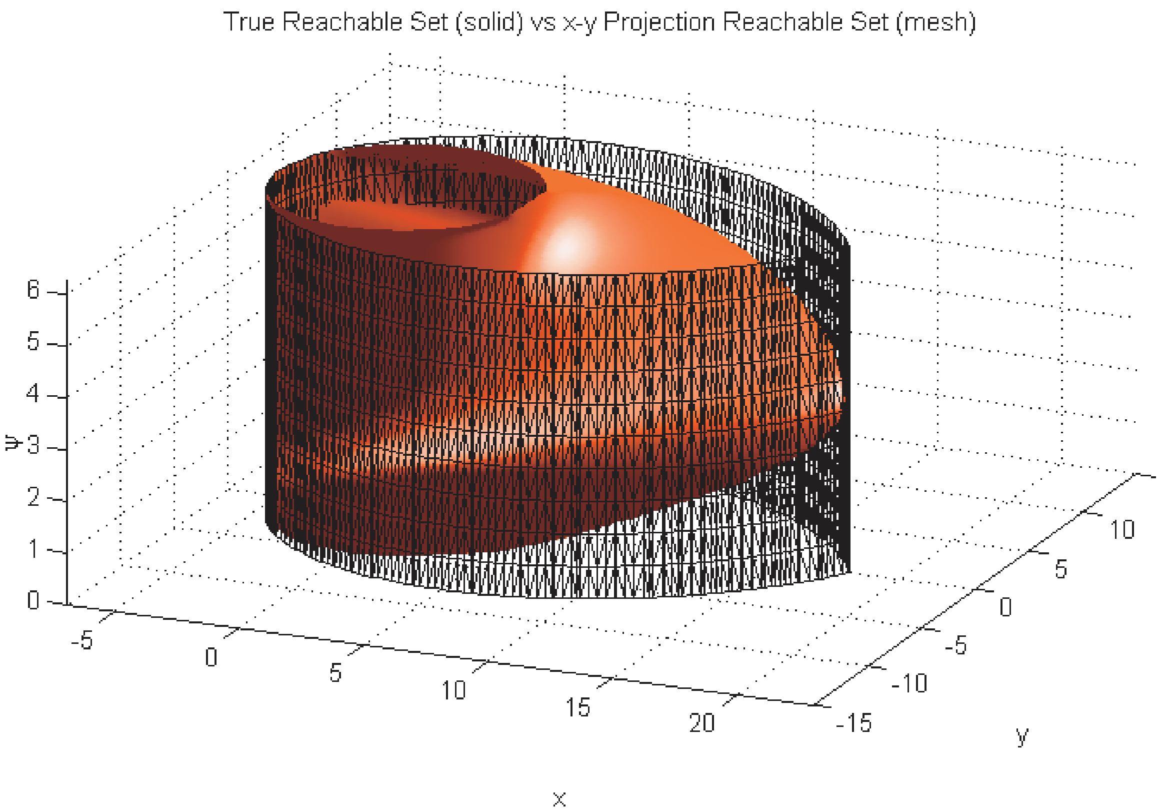
- What value to give free variable y in $f_i(x,y,z)$?
  - Treat it as a disturbance, bounded by the other projections

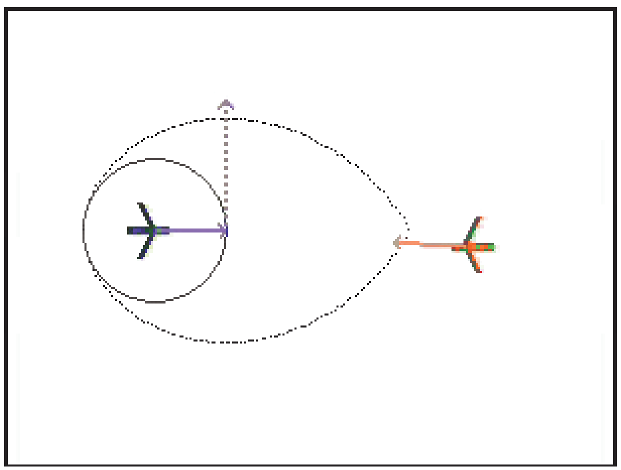
$$D_t\phi_{xz}(x,y,z,t) + \min_y \left[p_1 f_1(x,y,z) + p_3 f_3(x,y,z)\right] = 0$$

- Hamiltonian no longer depends on y, so computation can be done entirely in x–z space on $\phi_{xz}(x,z,t)$

Projective Collision Avoidance

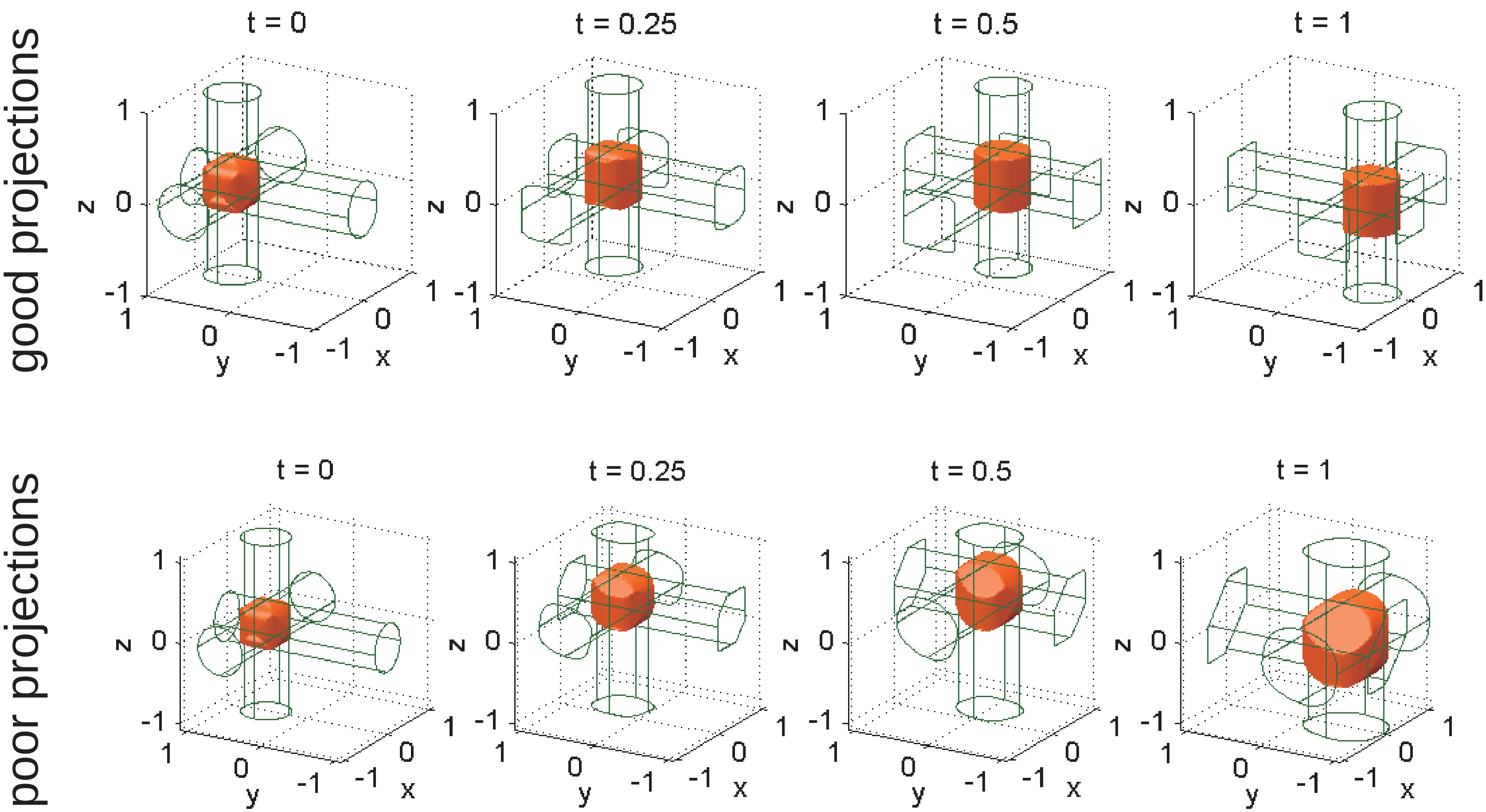
- Work strictly in relative x–y plane
  - Treat relative heading $\psi \in [0, 2\pi]$ as a disturbance input
  - Compute time: 40 seconds in 2D vs 20 minutes in 3D
  - Compare overapproximative prism (mesh) to true set (solid)





Projection Choices

- Poorly chosen projections may lead to large overapproximations
  - Projections need not be along coordinate axes
  - Number of projections is not constrained by number of dimensions



Hybrid System Reach Sets

Combining Continuous and Discrete Evolution

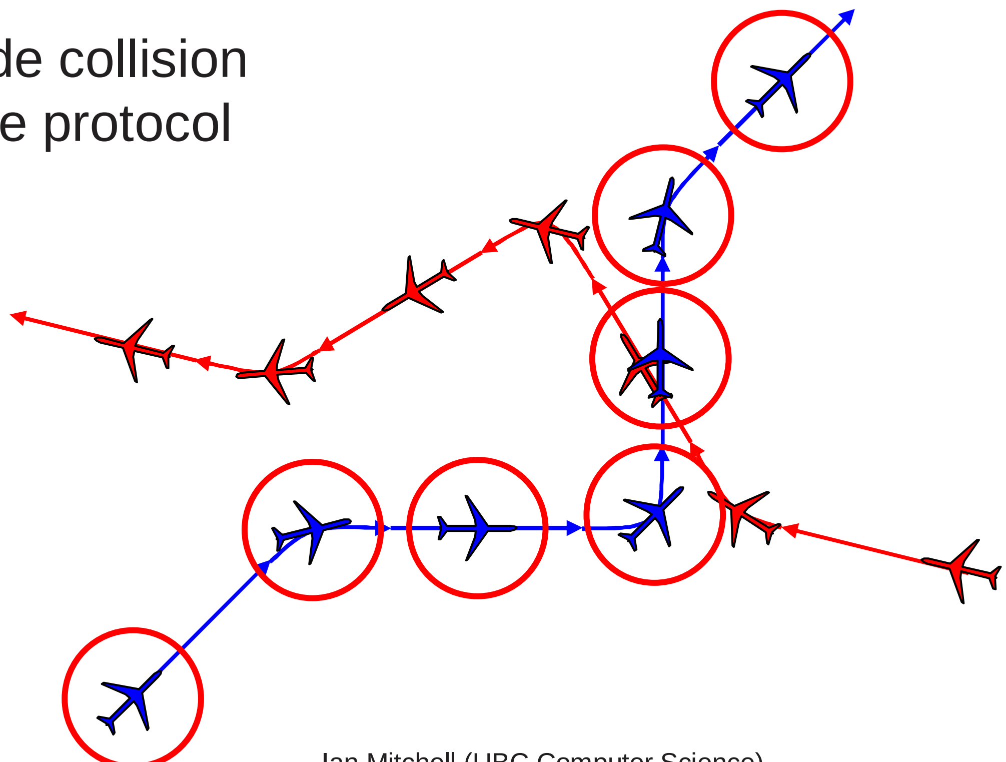
Outline

- Hybrid System example
  - Seven mode collision avoidance and results
- Hybrid Reachability
  - Implementing the reach-avoid operator
- Example applications
  - Discrete abstraction
  - Display analysis
  - Autolander

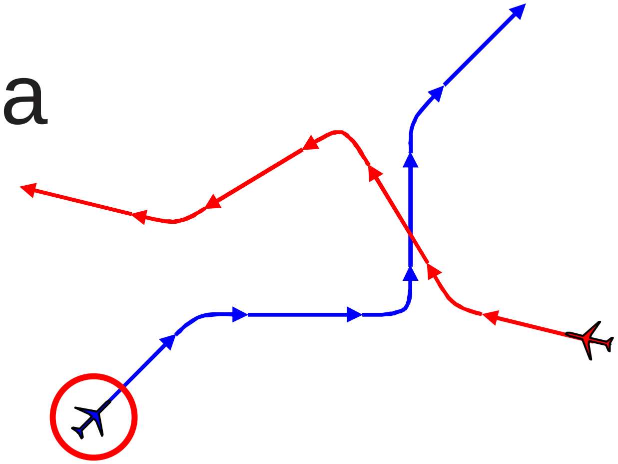
Why Hybrid Systems?

- Computers are increasingly interacting with external world
  - Flexibility of such combinations yields huge design space
  - Design methods and tools targeted (mostly) at either continuous or discrete systems
- Example: aircraft flight control systems

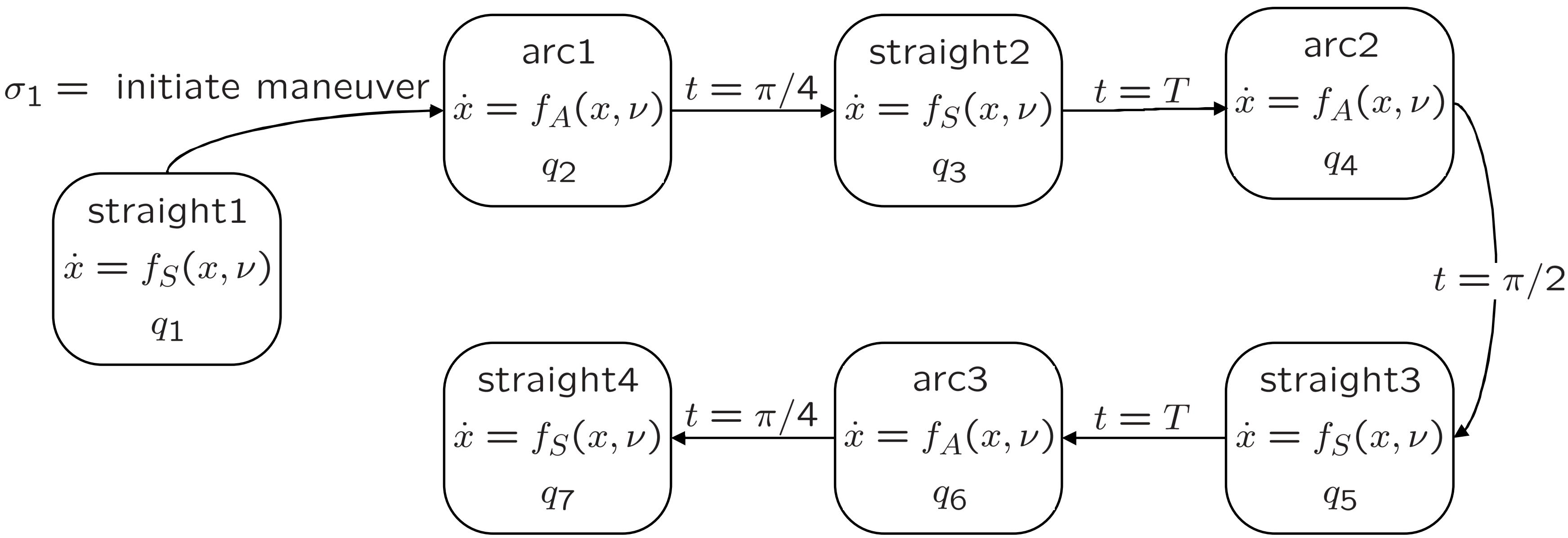
seven mode collision
avoidance protocol



Hybrid Automata



- Discrete modes and transitions
- Continuous evolution within each mode



$$f_S \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -v + v\cos\psi \\ v\sin\psi \end{bmatrix}$$

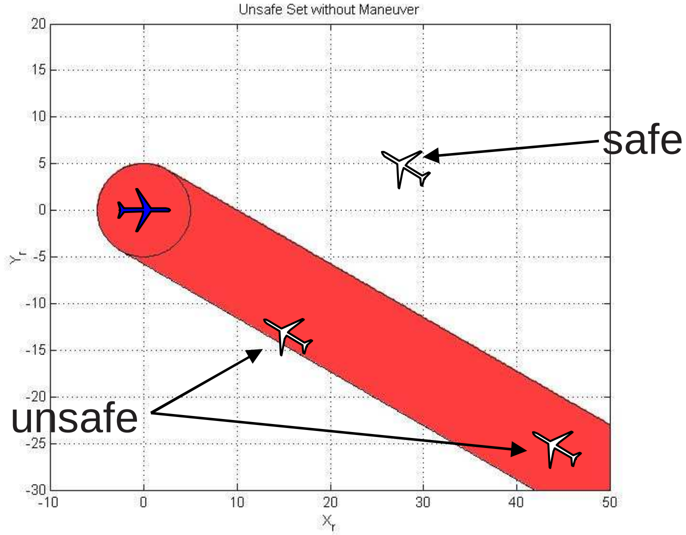
dynamics in straight modes

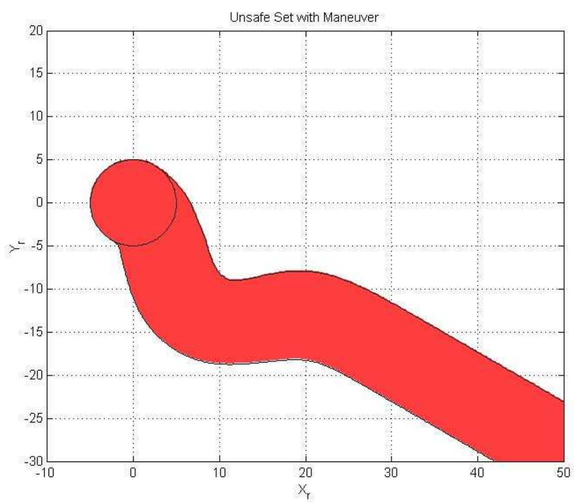
$$f_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -v + v\cos\psi - x_2 \\ v\sin\psi + x_1 \end{bmatrix}$$

dynamics in arc modes

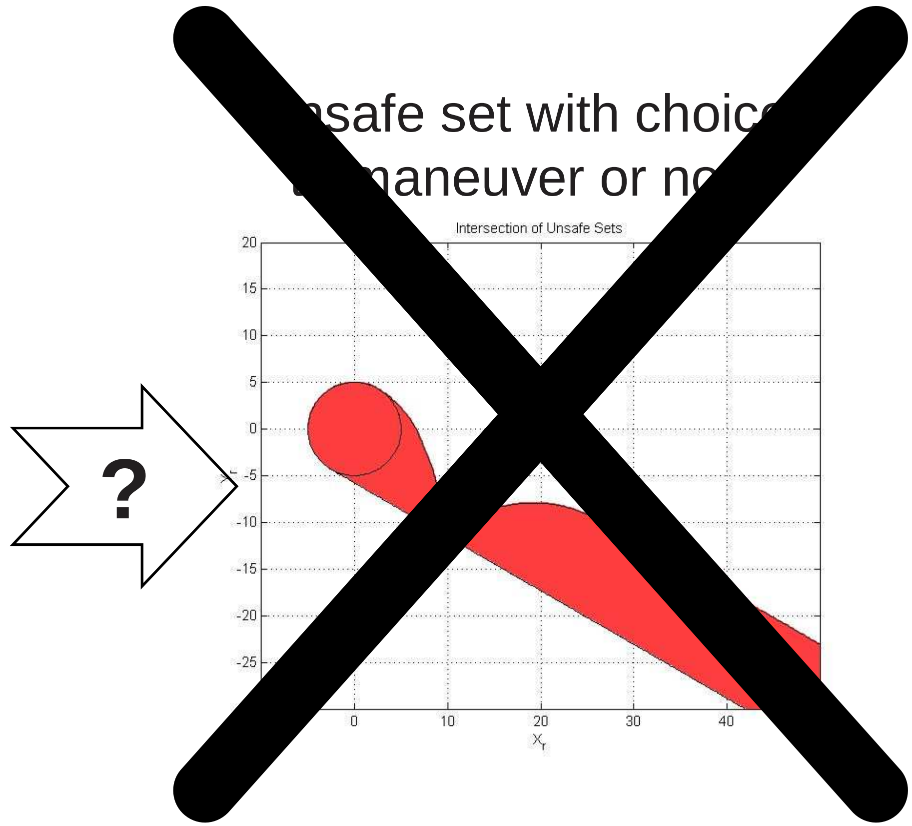
Seven Mode Safety Analysis

unsafe set without maneuver



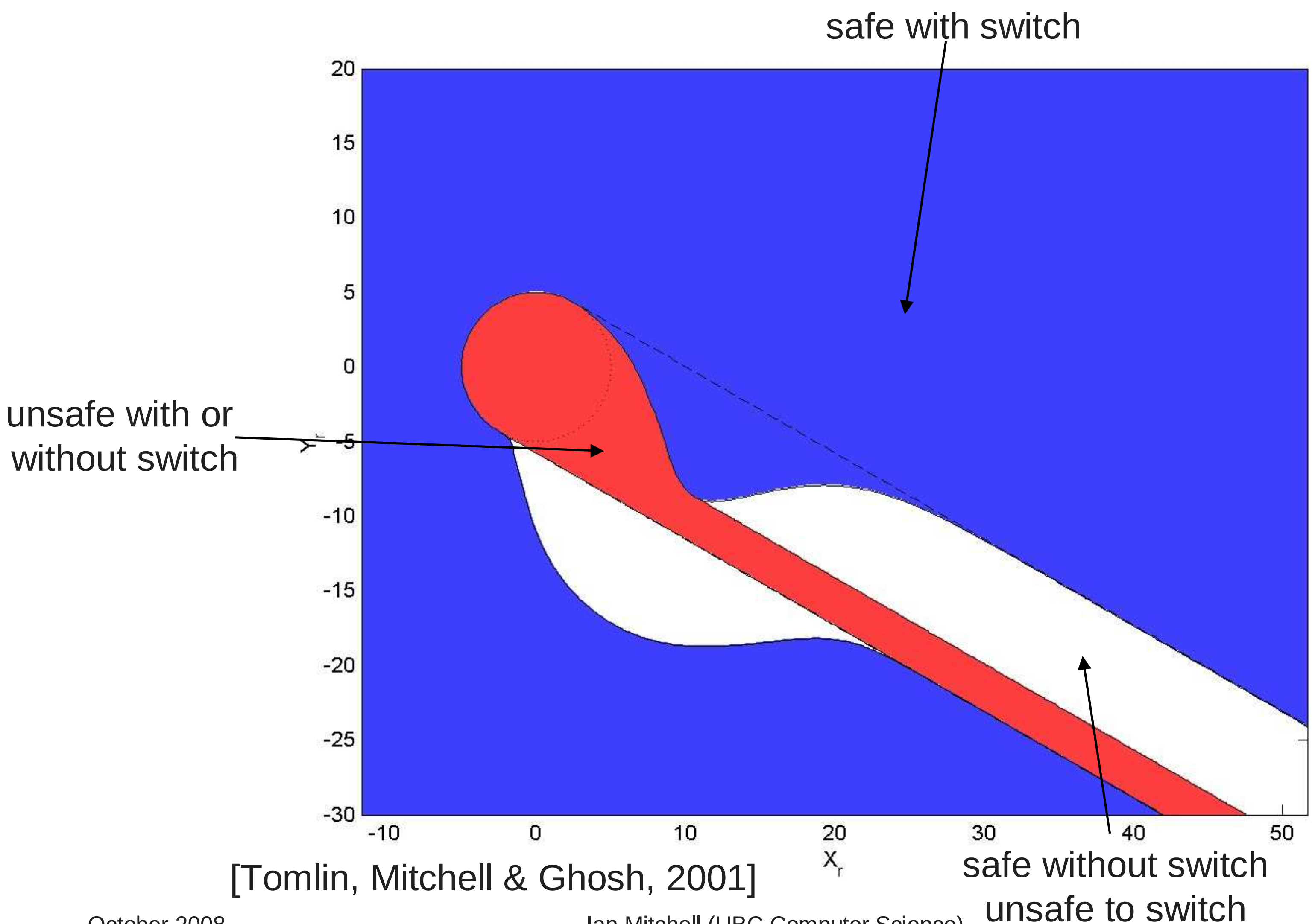


unsafe set with maneuver



Seven Mode Safety Analysis

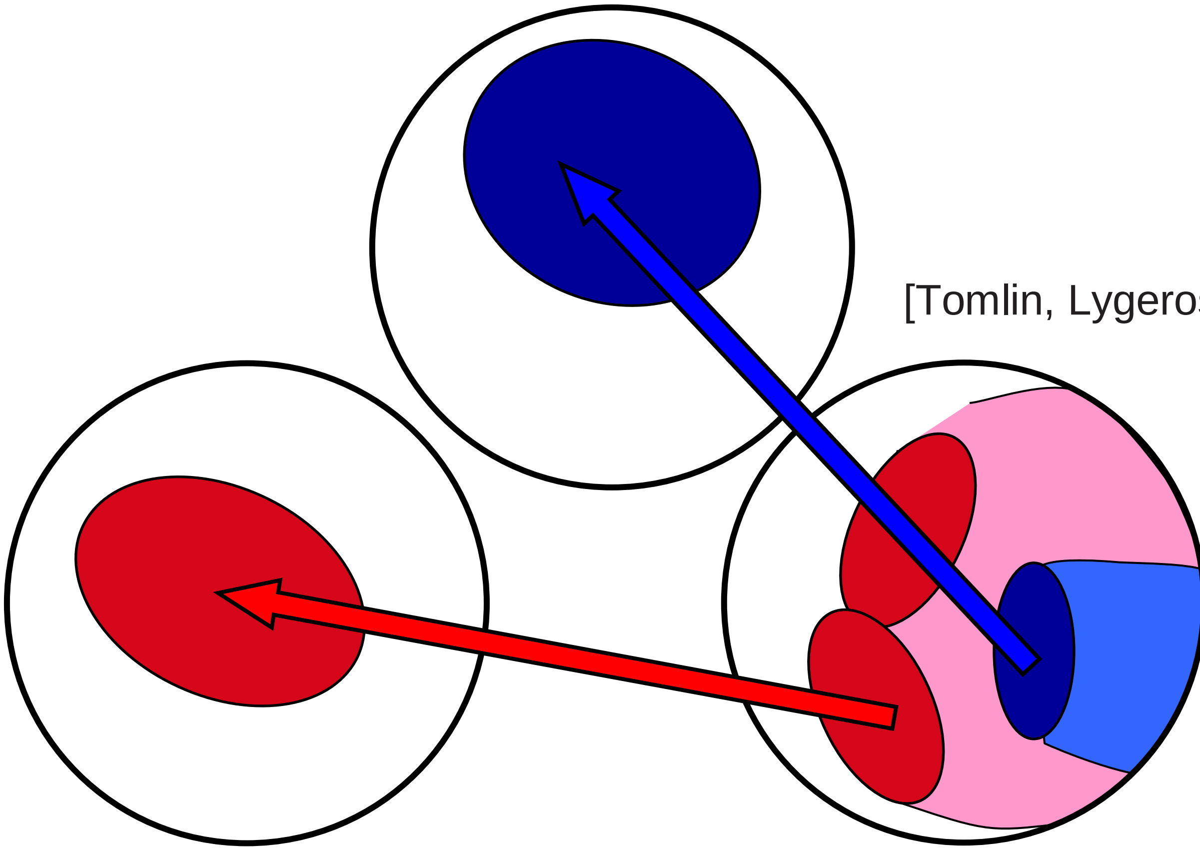
- Ability to choose maneuver start time further reduces unsafe set



[Tomlin, Mitchell & Ghosh, 2001]

Computing Hybrid Reachable Sets

- Compute continuous reachable set in each mode separately
  - Uncontrollable switches may introduce unsafe sets
  - Controllable switches may introduce safe sets
  - Forced switches introduce boundary conditions

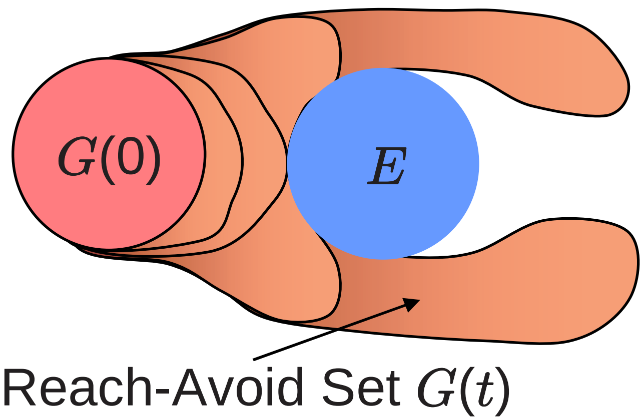


[Tomlin, Lygeros & Sastry, 2000]

Reach-Avoid Operator

- Compute set of states which reaches $G(0)$ without entering E

$$G(t) = \{x \in \mathbb{R}^n \mid \phi_G(x,t) \leq 0\}$$
$$E = \{x \in \mathbb{R}^n \mid \phi_E(x) \leq 0\}$$



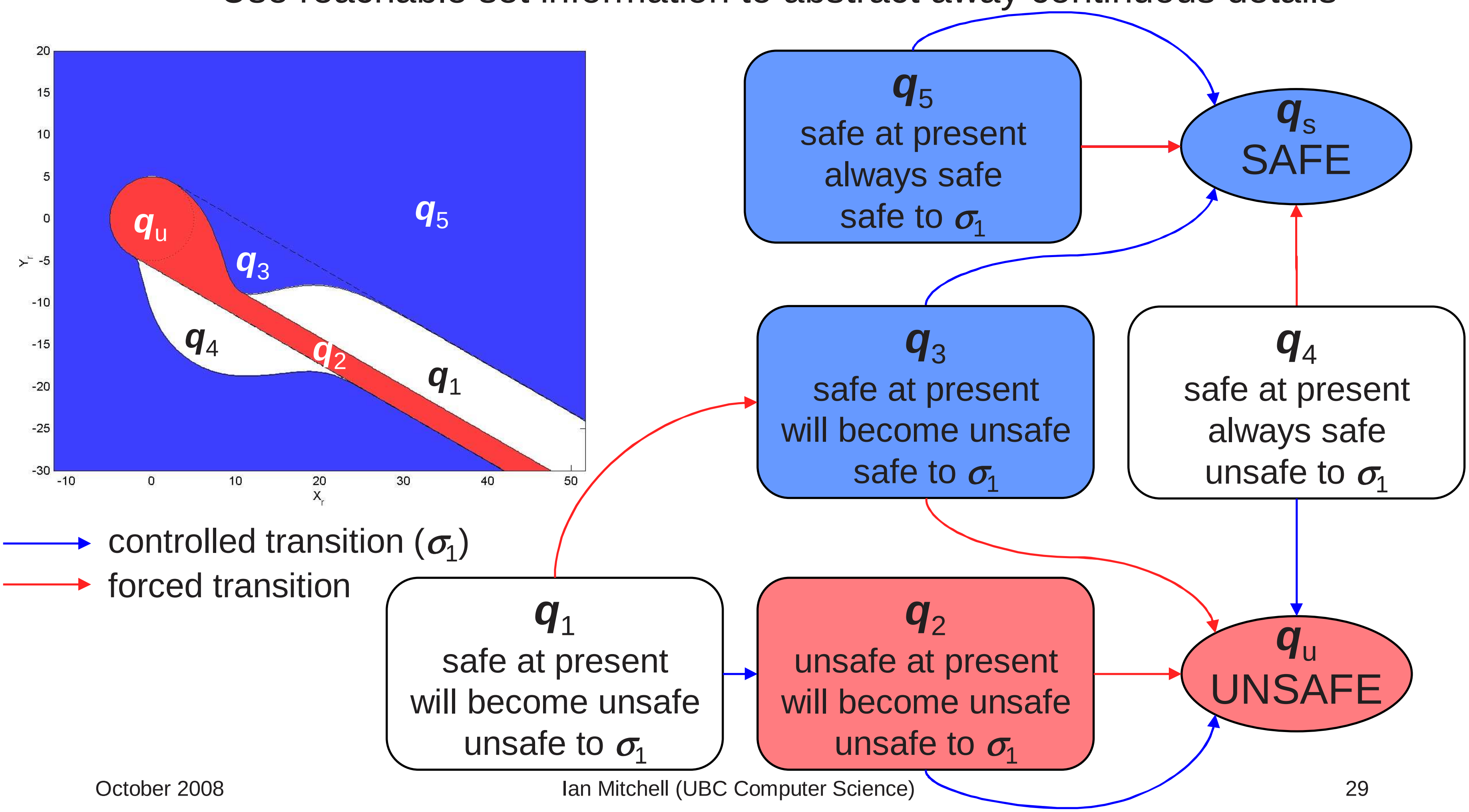
- Formulated as a constrained Hamilton-Jacobi equation or variational inequality
  - [Mitchell & Tomlin, 2000]

$$D_t\phi_G(x,t) + \min\left[0, H(x, D_x\phi_G(x,t))\right] = 0$$
$$\text{subject to: } \phi_G(x,t) \geq \phi_E(x)$$

- Level set can represent often odd shape of reach-avoid sets

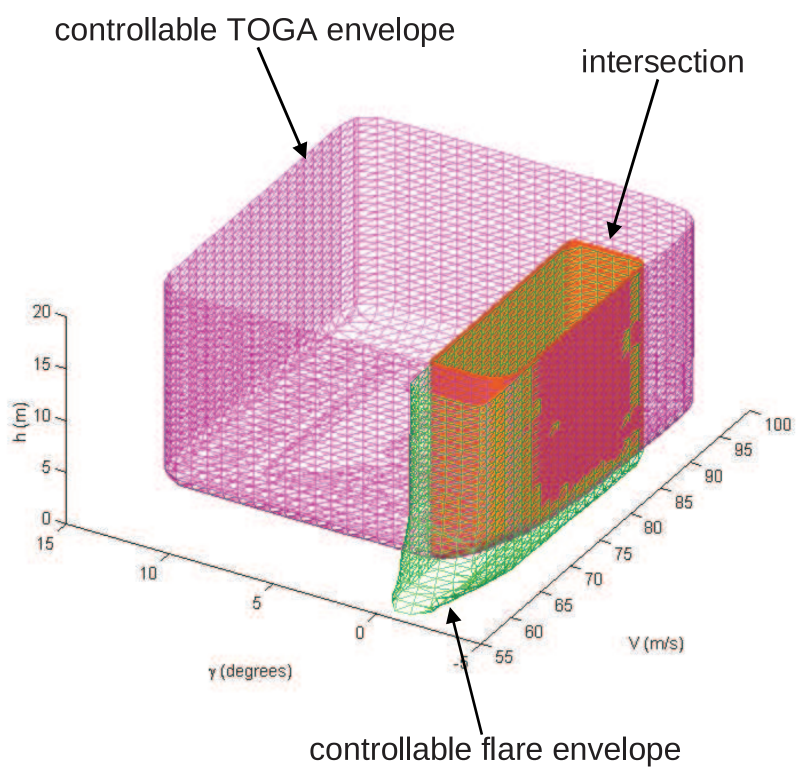
Application: Discrete Abstractions

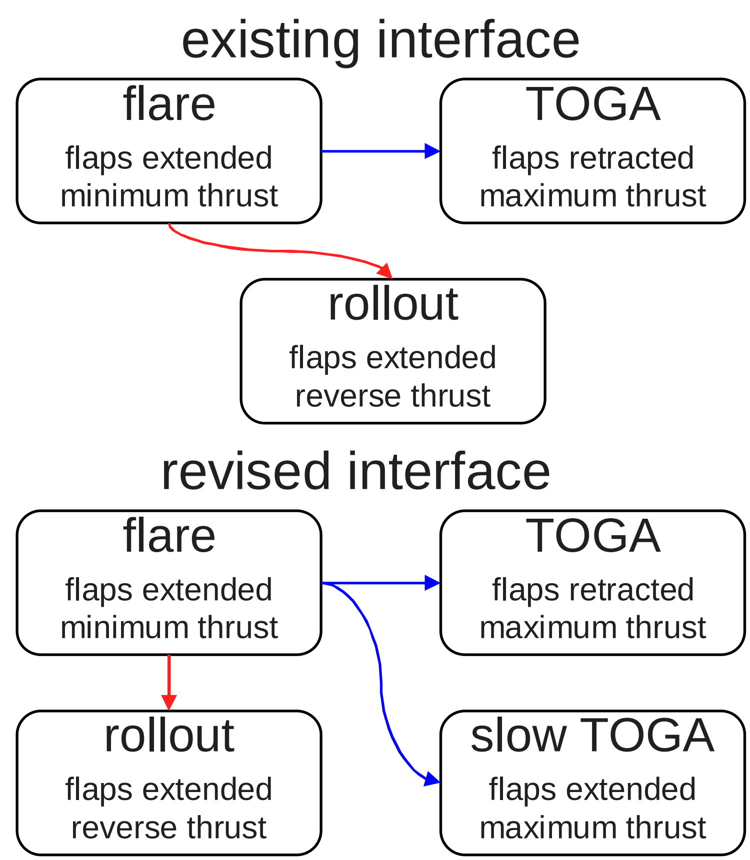
- It can be easier to analyze discrete automata than hybrid automata or continuous systems
  - Use reachable set information to abstract away continuous details



Application: Cockpit Display Analysis

- Controllable flight envelopes for landing and Take Off / Go Around (TOGA) maneuvers may not be the same
- Pilot's cockpit display may not contain sufficient information to distinguish whether TOGA can be initiated

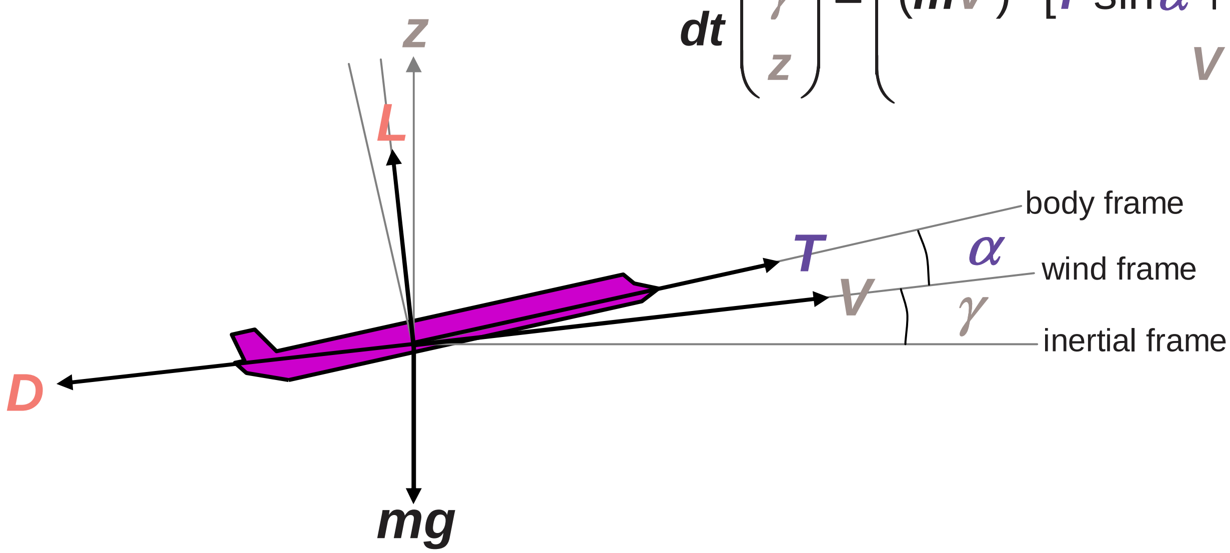




Application: Aircraft Autolander

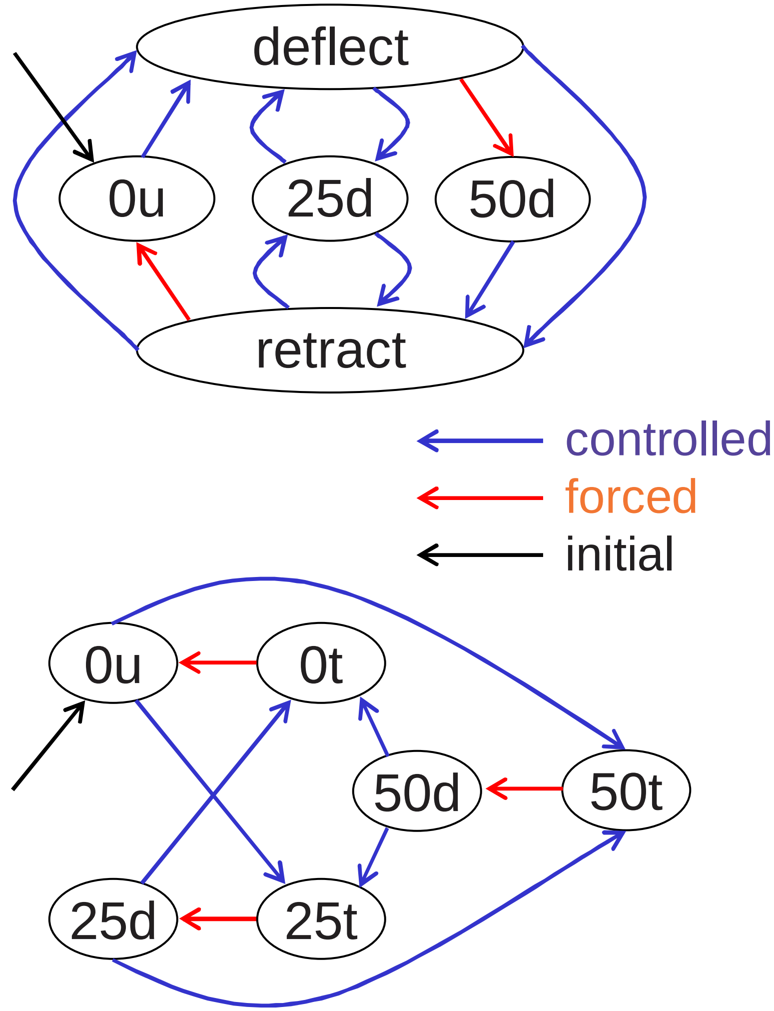
- Airplane must stay within safe flight envelope during landing
  - Bounds on velocity (V), flight path angle (γ), height (z)
  - Control over engine thrust (T), angle of attack (α), flap settings
  - Model flap settings as discrete modes of hybrid automata
  - Terms in continuous dynamics may depend on flap setting
  - [Mitchell, Bayen & Tomlin, 2001]

$$\frac{d}{dt}\begin{pmatrix} V \\ \gamma \\ z \end{pmatrix} = \begin{pmatrix} m^{-1}[T\cos\alpha - D(\alpha, V) - mg\sin\gamma] \\ (mV)^{-1}[T\sin\alpha + L(\alpha, V) - mg\cos\gamma] \\ V\sin\gamma \end{pmatrix}$$



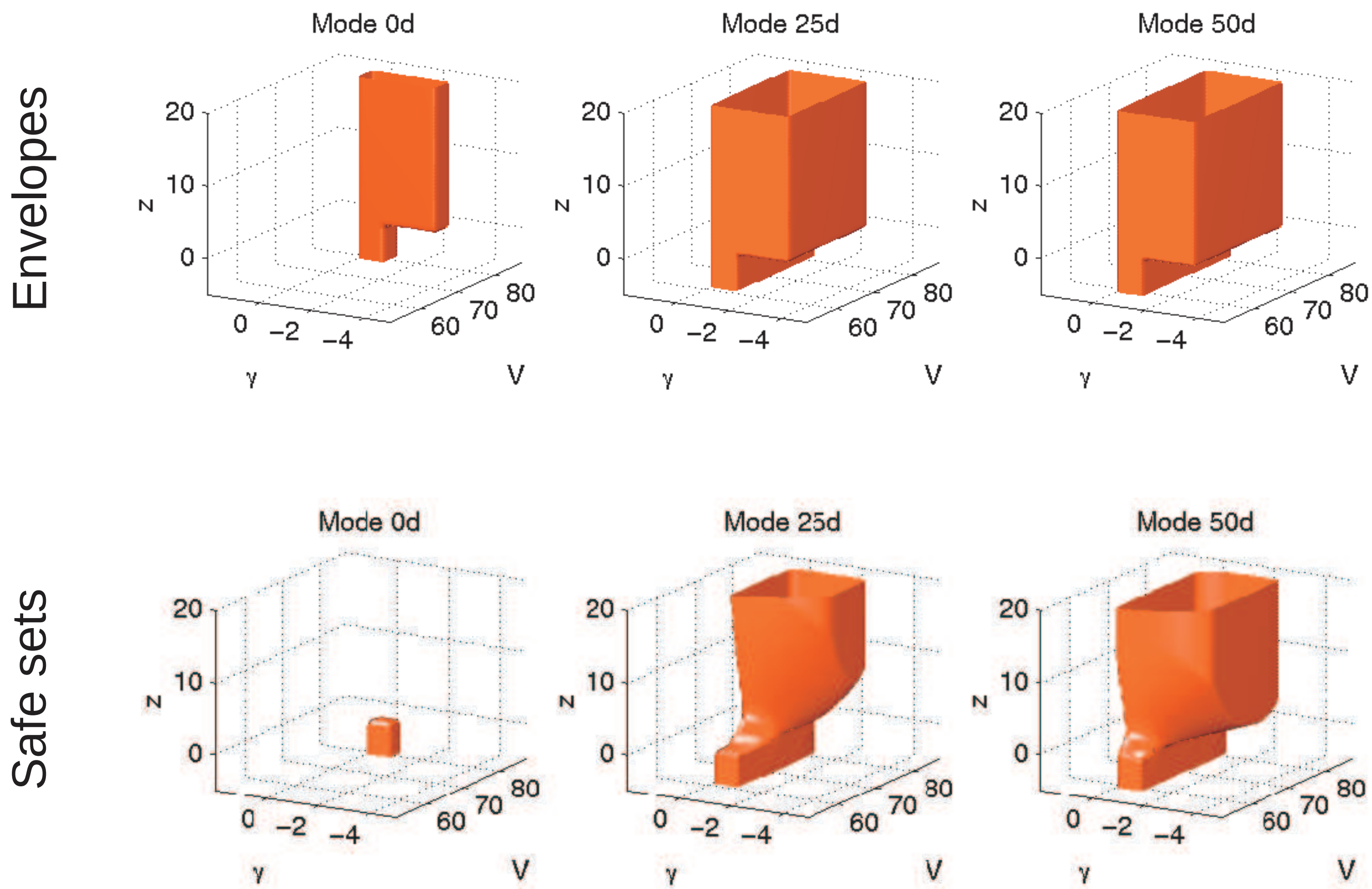
Landing Example: Discrete Model

- Flap dynamics version
  - Pilot can choose one of three flap deflections
  - Thirty seconds for zero to full deflection

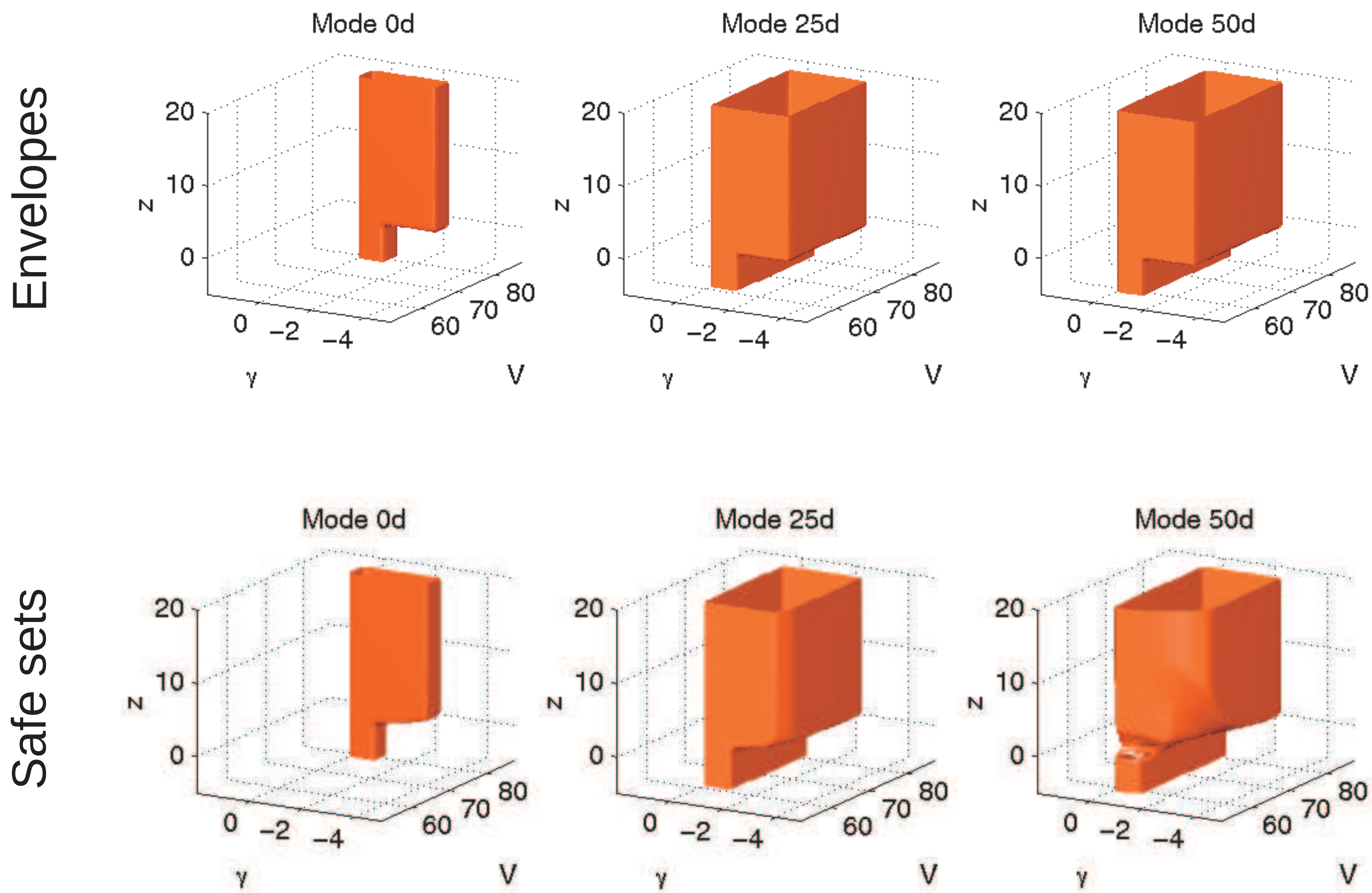


- Implemented version
  - Instant switches between fixed deflections
  - Additional timed modes to remove Zeno behavior

Landing Example: No Mode Switches

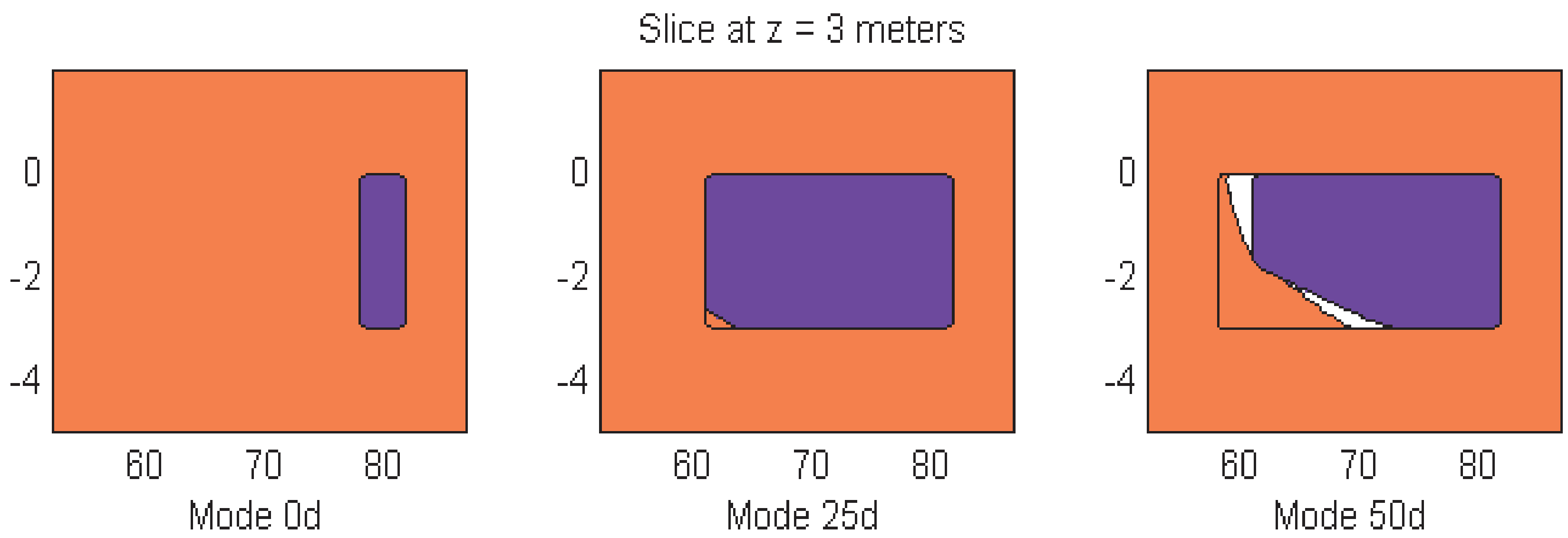


Landing Example: Mode Switches



Landing Example: Synthesizing Control

- For states at the boundary of the safe set, results of reach-avoid computation determine
  - What continuous inputs (if any) maintain safety
  - What discrete jumps (if any) are safe to perform
  - Level set values & gradients provide all relevant data



Viability Theory

An Alternative Approach Based on Set Valued Analysis

Outline

- Differential inclusions
- Constructs from viability
  - Capture Basin
  - Viability Kernel
- The contingent cone
- Defining the viability kernel
- Approximating the viability kernel

Differential Inclusions

- Dynamics defined by differential inclusion

$$\frac{dx}{dt} \in \mathcal{F}(x), \qquad \mathcal{F}(x) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$$

  - For example

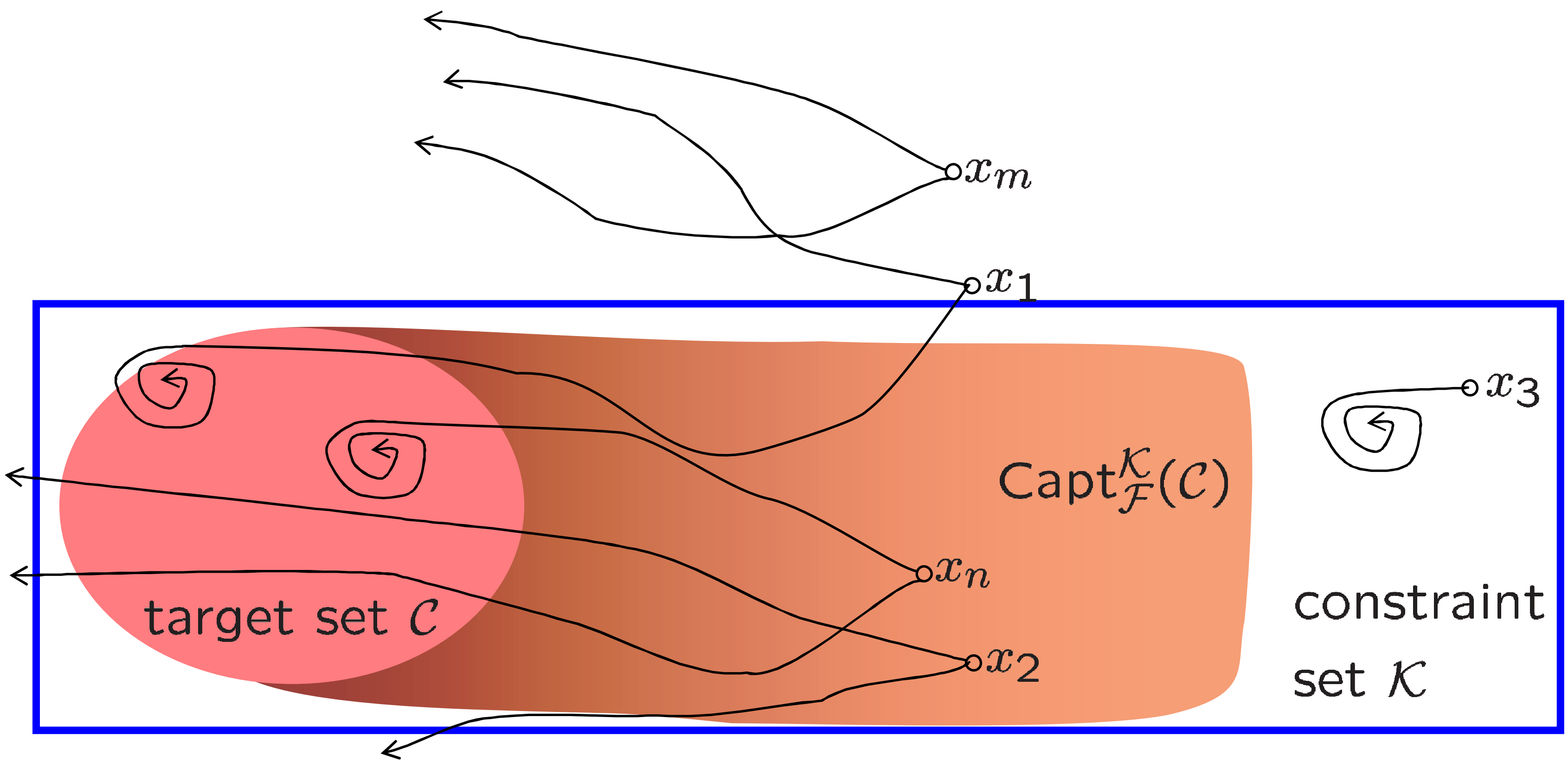
$$\mathcal{F}(x) = \{y \in \mathbb{R}^n \mid \exists b \in \mathcal{B}, y = f(x, b)\}$$

- Set-valued map $\mathcal{F}(x)$ has Lipschitz-like but less restrictive conditions
  - For example, discontinuous $f(x,b)$ can be represented
- Extensions exist for differential game settings

Capture Basin

$$\text{Capt}_{\mathcal{F}}^{\mathcal{K}}(\mathcal{C}) = \{x_i \mid \exists t > 0, x(t) \in \mathcal{C} \wedge \forall s \in [0,t], x(s) \in \mathcal{K}\}$$

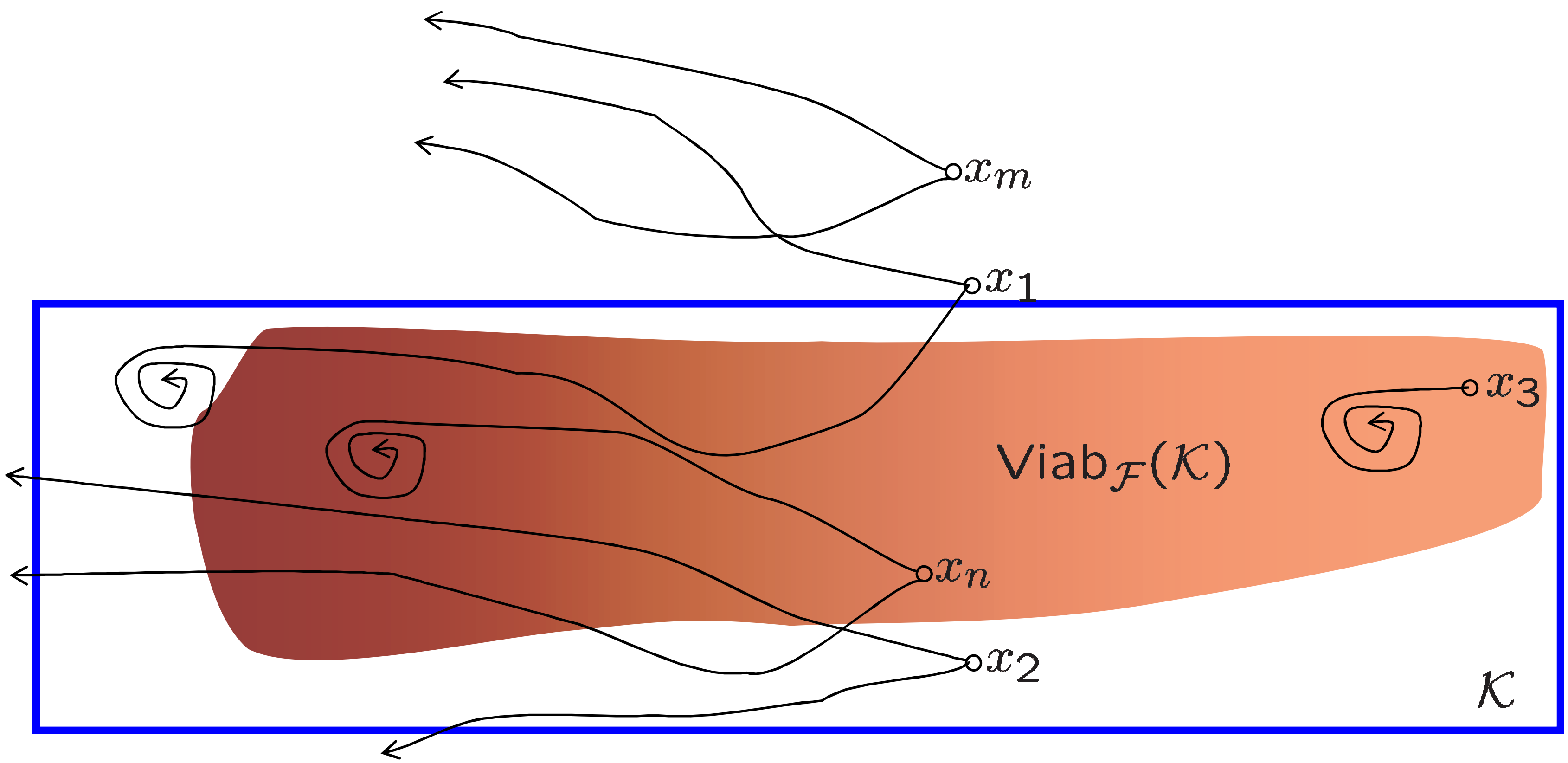
$$\text{where } \dot{x} \in \mathcal{F}(x) \text{ and } x(0) = x_i$$



Viability Kernel

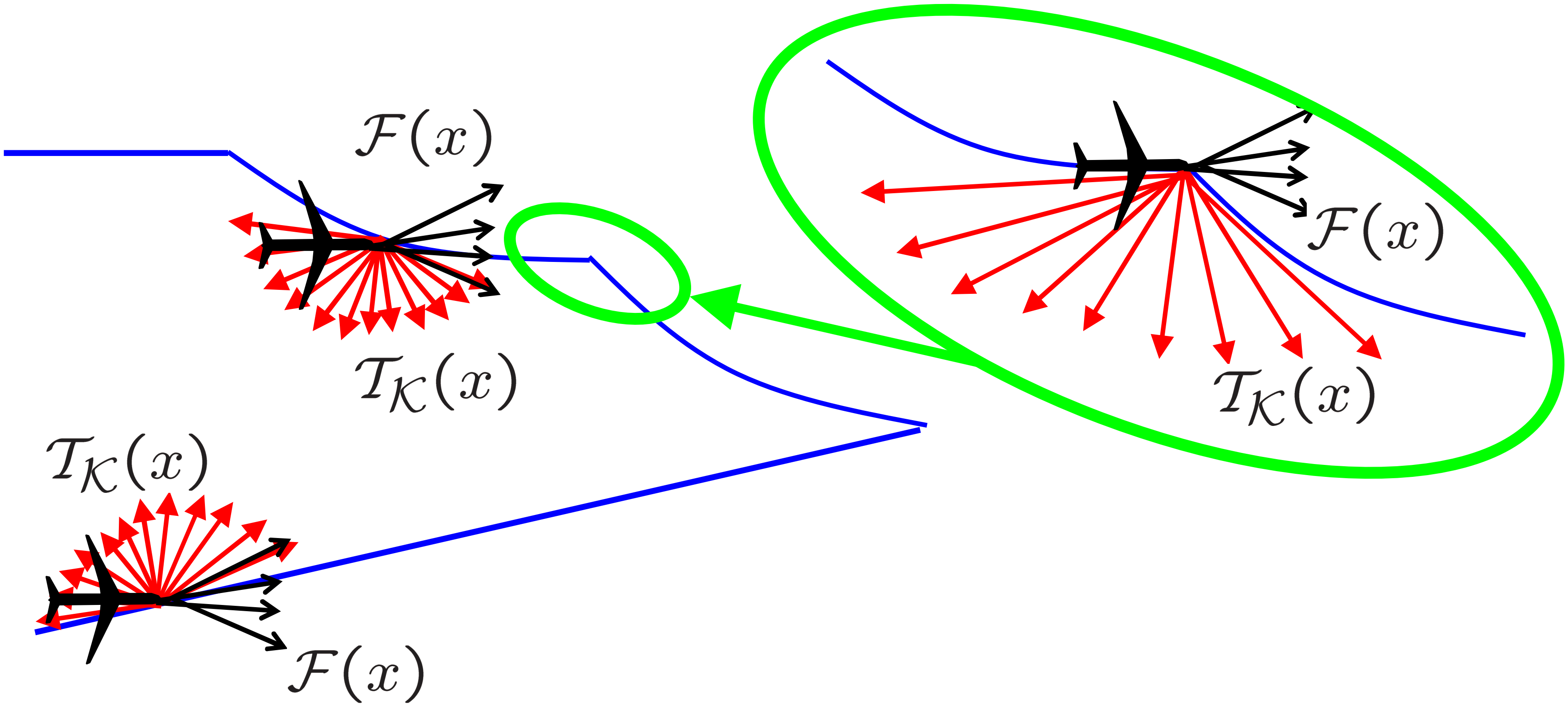
$$\text{Viab}_{\mathcal{F}}(\mathcal{K}) = \{x_i \mid \forall t > 0, x(t) \in \mathcal{K}\}$$

$$\text{where } \dot{x} \in \mathcal{F}(x) \text{ and } x(0) = x_i$$



The Contingent Cone

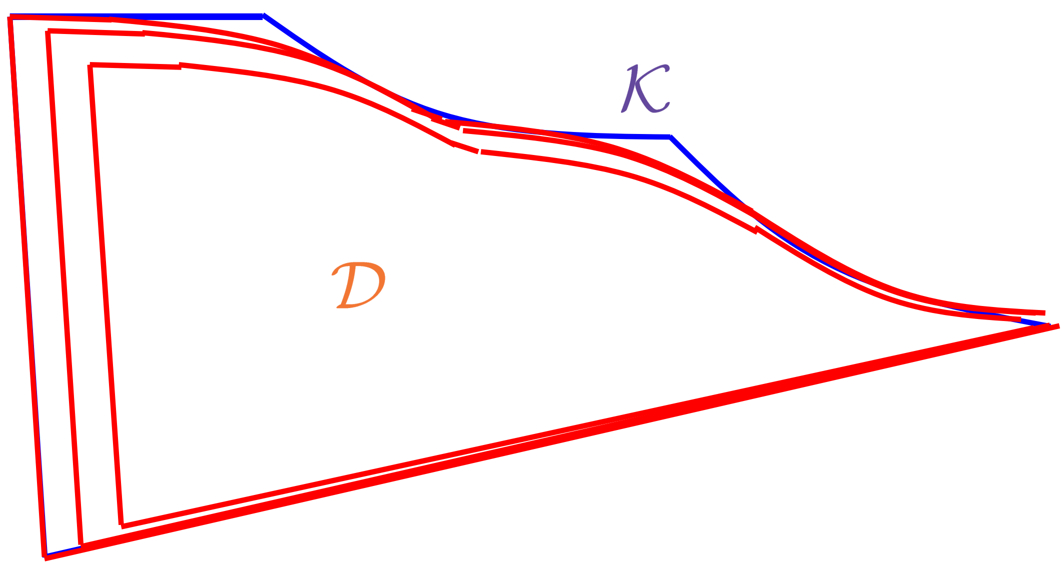
$$\mathcal{T}_{\mathcal{K}}(x) = \limsup_{h \to 0^+} \frac{\mathcal{K} - x}{h}$$



Defining the Viability Kernel

Assume $\mathcal{F}$ is Marchaud and $\mathcal{K}$ is closed.

Then $\text{Viab}_{\mathcal{F}}(\mathcal{K})$ is the largest closed $\mathcal{D}$ such that $\mathcal{F}(x) \cap \mathcal{T}_{\mathcal{D}}(x) \neq \emptyset$.



$$\mathcal{F} \text{ is Marchaud if } \begin{cases} \text{the graph and domain of } \mathcal{F} \text{ are nonempty and closed,} \\ \text{the values } \mathcal{F}(x) \text{ are convex,} \\ \text{the growth of } \mathcal{F} \text{ is linear: } \exists c > 0, \forall x, \sup\limits_{v \in \mathcal{F}(x)} \|v\| \leq c(\|x\| + 1) \end{cases}$$

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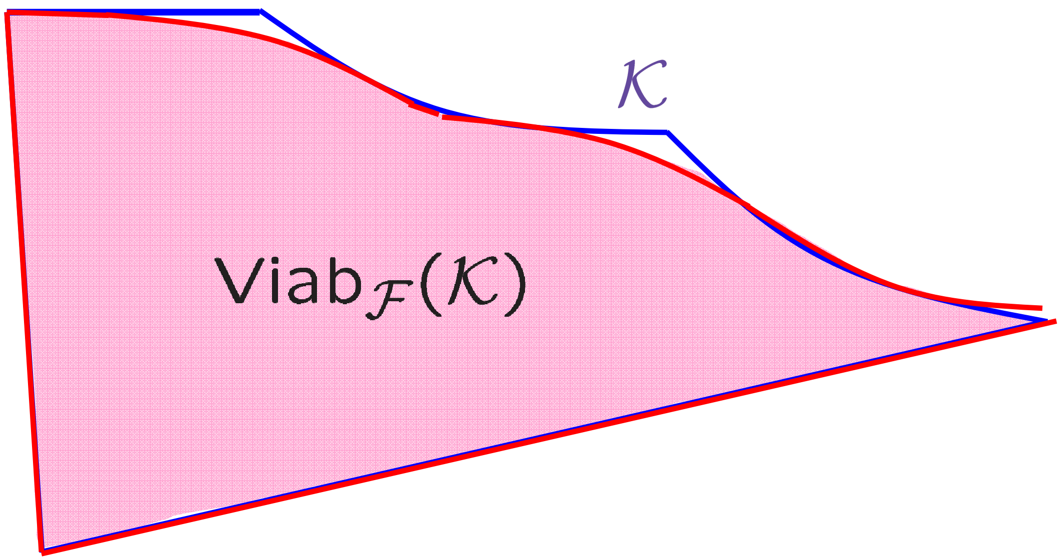

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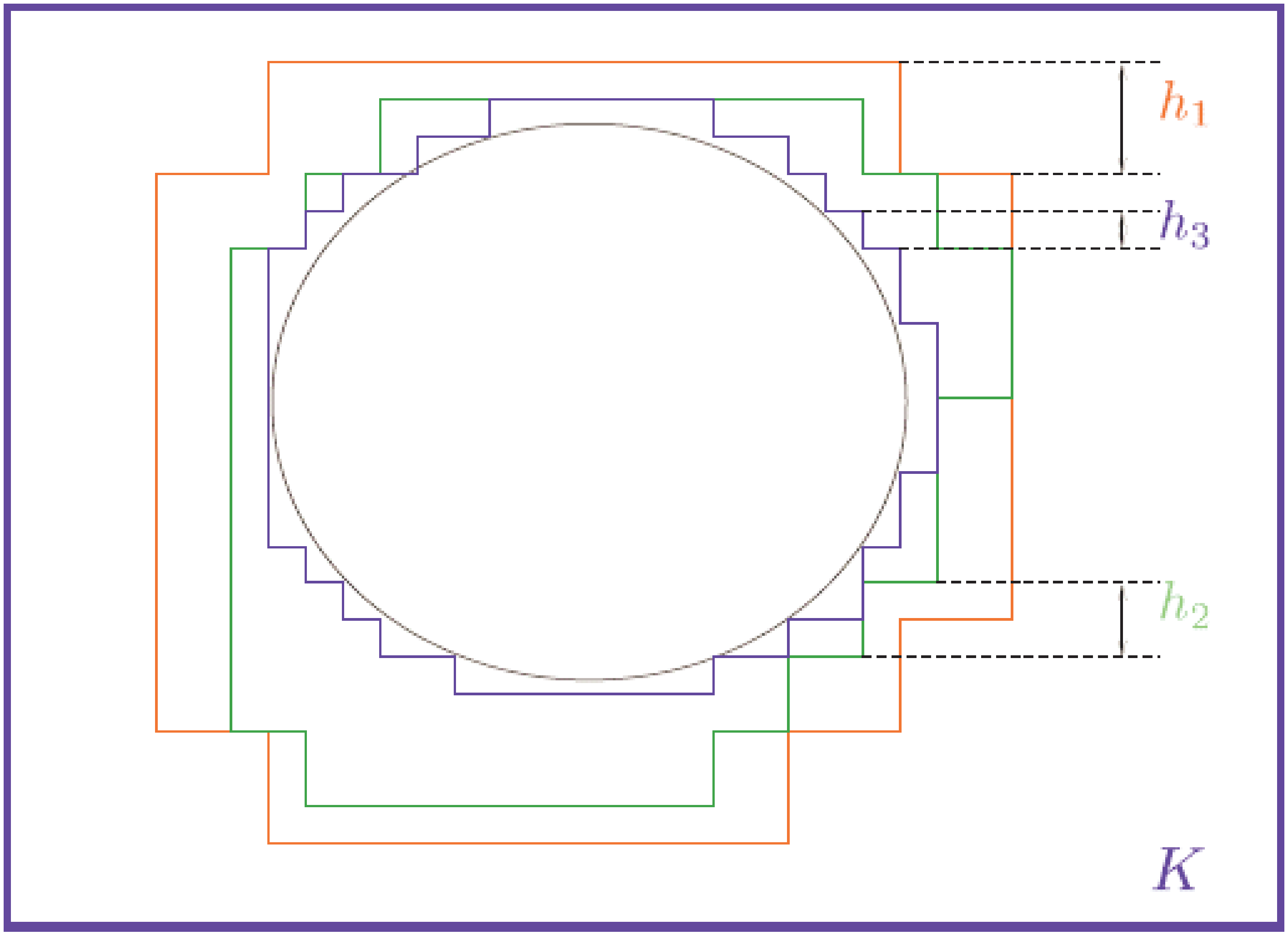
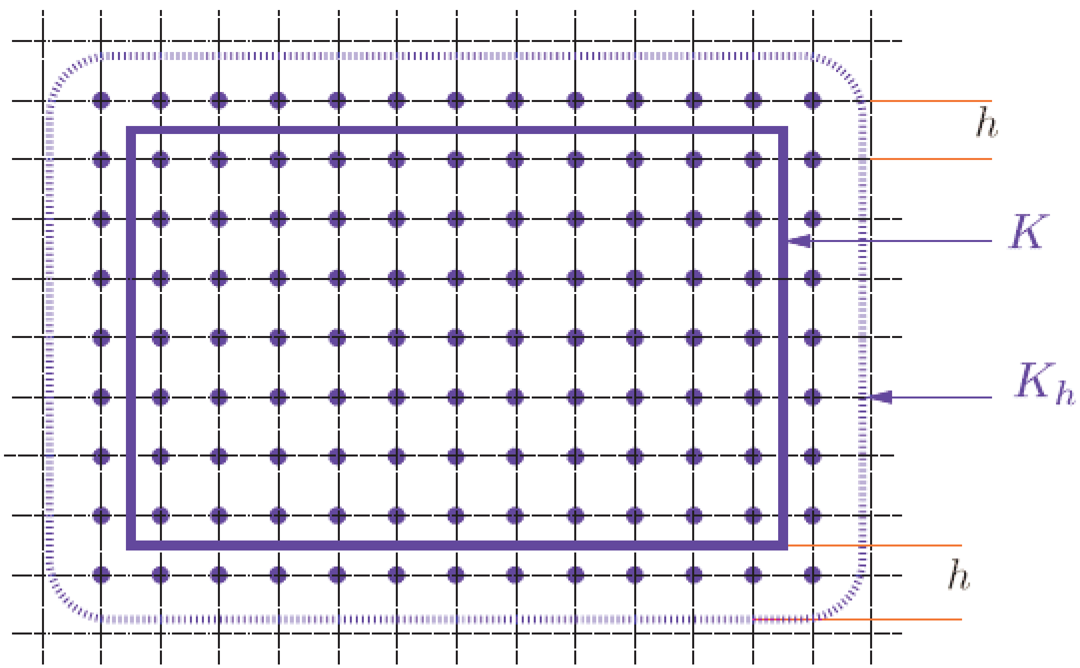
Approximating the Viability Kernel

Algorithm will perform systematic outer approximation of the reachable set at various discretization levels



Discretization of the Constraint Set

Discretization of space	h
Discretization of time	ρ
Lipschitz constant of dynamics $F(x)$:	l, w.r.t. x
Maximal magnitude of dynamics $F(x)$:	M

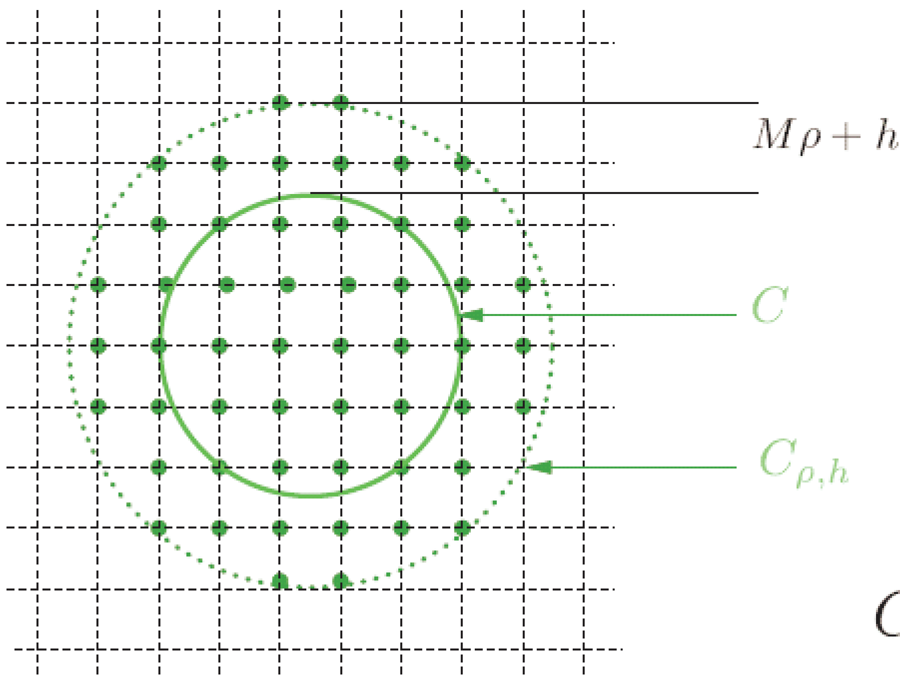


$$X_h = h\mathbb{N}^N$$

$$K_h = (K + h\mathcal{B}) \cup X_h$$

Discretization of the Target for Capture Basins

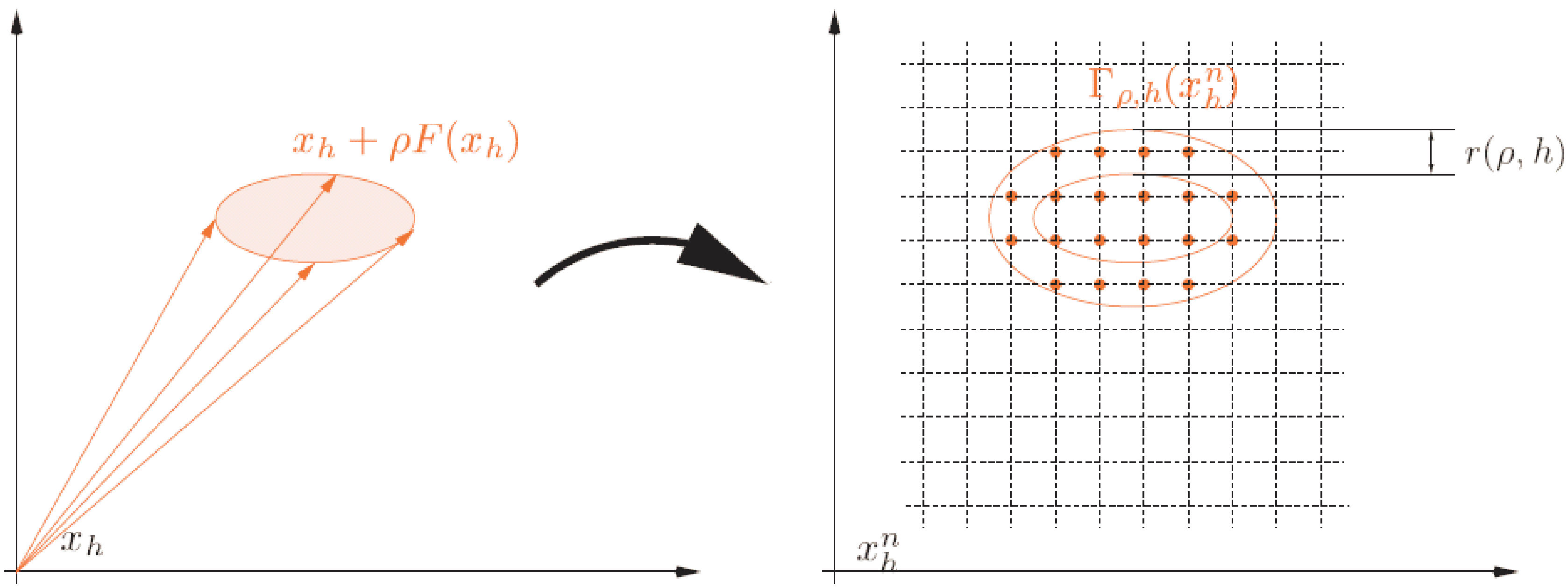
Discretization of space	h
Discretization of time	ρ
Lipschitz constant of dynamics $F(x)$:	l, w.r.t. x
Maximal magnitude of dynamics $F(x)$:	M



$$C_{\rho,h} = (C + (M\rho + h)\mathcal{B}) \cup X_h$$

Discretization of the Dynamics

$$x_h^{n+1} \in \Gamma_{\rho,h}(x_h^n) := [x_h^n + \rho\,(\,F(x_h^n) + r(\rho,h)\mathcal{B}\,)] \cap X_h$$



Dilation factor $r(\rho,h) = lh + Ml\rho + 2\frac{h}{\rho}$

CFL-type condition $\rho = \sqrt{\frac{h}{Ml}}$

Discrete Viability Algorithm

- Apply discrete viability algorithm on discretized dynamics for discretized constraint set

$$K_h^0 = K_h,$$
$$K_h^{n+1} = \{x \in K_h^n \mid \Gamma_{\rho,h}(x) \cap K_h^n \neq \emptyset\}.$$

  – Approximation will reach a fixed point after finite iterations
  – Approximation (plus slight dilation) will contain true viability kernel
  – Approximation will converge to true viability kernel as discretization parameters go to zero
- Many algorithmic refinements to improve efficiency
  – Efficient construction of next iteration
  – Grid refinement without starting from scratch
- Details in [Cardaliaguet, Quincampoix & Saint-Pierre, “Set-valued numerical analysis for optimal control and differential games” in *Stochastic and Differential Games: Theory and Numerical Methods* (Bardi, Raghavan & Parthasarathy, eds.), Birkhäuser, pp. 177–247 (1999)]