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Eigenvalue bounds for saddle-point systems with singular leading blocks

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ABSTRACT

We derive bounds on the eigenvalues of saddle-point matrices with singular leading blocks. The technique of proof is based on augmenting the singular leading block to replace it with a positive definite matrix. Our bounds depend on the principal angles between the ranges or kernels of the matrix blocks. Numerical experiments validate our analytical findings.

1. Introduction

Consider the saddle-point system

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite and $B \in \mathbb{R}^{m\times n}$ has full row rank, with $m < n$. We denote the coefficient matrix by

$$\mathcal{K} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}. \tag{2}$$

We assume throughout that $\mathcal{K}$ is invertible. Our goal in this paper is to derive eigenvalue bounds for $\mathcal{K}$ under the assumption that A is singular.

Matrices of the form (2) with a singular A arise in several applications. Examples include: the time-harmonic Maxwell equations [1]; constrained weighted least-squares [2, sec.2.2]; geophysical inverse problems [3]; dual–dual finite element formulations [4,5]; boundary element tearing and interconnecting methods [6]; the Darcy–Stokes equations [7]; and some finite element formulations of the Stokes equations [8,9]. See, for example, [10, Chapter 2] for detailed formulations and additional discussion.

Related work. The eigenvalues of saddle-point matrices have been considered in a variety of papers, under different assumptions on the blocks A and B. A seminal paper in this area is that of Rusten and Winther [11]. They assume that the leading block A is positive definite. Their eigenvalue bounds still apply when A is singular (as we discuss in more detail in Section 2), but in that case their lower positive eigenvalue bound is impractical, as it reduces to zero.

A refinement of the lower positive eigenvalue bound of Rusten and Winther is proposed by Ruiz, Sartenaer, and Tannier [12]. As in the case of Rusten and Winther, they assume A is positive definite but their analyses do not require

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this, and extend to the case when A is singular. Their bound is more intricate and is generally nonzero, even for singular A (with some exceptions; we discuss their bound in more detail in Section 3.3). However, the analysis of Ruiz et al. requires orthonormality of the columns of B. As pointed out in their paper, this can be achieved by preconditioning, but means that the applicability of their bound is rather limited. It applies only to preconditioned matrices that satisfy this rather stringent orthonormality requirement, and does not generally apply to unpreconditioned saddle-point matrices.

Spectral properties of saddle-point matrices have also been considered for matrices with indefinite leading blocks [13]; matrices with a stabilization term in the (2,2)-block [14]; and in matrices arising from specific applications, such as interior point methods [15,16]. Recent papers have also generalized some of these results to block-3×3 or block-$n \times n$ multiple saddle-point systems; see, for example, [17–19].

Contribution of this paper. We derive a lower bound on the positive eigenvalues of $\mathcal{K}$ that does not require invertibility of A, by considering the principal angles between the ranges/kernels of A and B, similar to what is done in Ruiz et al. [12]. Our analysis removes some of the restrictions of other works: in particular, we do not have any requirements on positive definiteness or eigenvalue scaling of A, and we do not require orthogonality of the columns of B.

Notation. Our analysis will rely on the eigenvalues and singular values of A and B, as well as some other matrices we will introduce later in the text. We will denote the eigenvalues of a matrix $M \in \mathbb{R}^{n\times n}$ by

$$\mu_i(M), \quad i = 1, \dots, n,$$

and in terms of ordering we will assume that

$$\mu_1(M) \ge \mu_2(M) \ge \cdots \ge \mu_n(M).$$

We follow the same convention for singular values of a rectangular matrix N, but we use σ rather than μ: i.e., the singular values of $N \in \mathbb{R}^{m\times n}$ are denoted by

$$\sigma_1(N) \ge \sigma_2(N) \ge \cdots \ge \sigma_m(N) \ge 0.$$

To increase clarity, we will often refer to the maximal eigenvalues/singular values $\mu_1(M)$ and $\sigma_1(N)$ by $\mu_{\max}(M)$ and $\sigma_{\max}(M)$ respectively. Similarly, we will refer to the minimal values $\mu_n(M)$, $\sigma_m(N)$ by $\mu_{\min}(M)$ and $\sigma_{\min}(M)$. The positive eigenvalues of a matrix will be denoted by a "+" superscript — for instance, we denote the smallest nonzero eigenvalue of a semidefinite matrix M by $\mu^+_{\min}(M)$.

Principal angles. Our analysis relies on the principal angles between certain subspaces. We include below a definition [20,21] based on the singular value decomposition (SVD):

Definition 1. Let the columns of matrices $X \in \mathbb{C}^{n\times p}$ and $Y \in \mathbb{C}^{n\times q}$ denote orthonormal bases of the subspaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $U\Sigma V^H$ denote the SVD of $X^H Y$, where U and V are unitary matrices and Σ is a $p \times q$ diagonal matrix with real diagonal elements $c_1, \dots, c_r$, with $r = \min(p, q)$. The singular values $c_1, \dots, c_r$ denote the cosines of the principal angles between $\mathcal{X}$ and $\mathcal{Y}$. The principal vectors associated with this pair of subspaces are given by the first r columns of XU and YV, correspondingly.

Outline. In Section 2 we discuss our general approach of augmenting the leading block of a saddle-point matrix to obtain a lower bound on the positive eigenvalues. In Section 3 we provide new bounds, which rely on the angles between the kernels of A and B. We then present numerical experiments in Section 4 and concluding remarks in Section 5.

2. Lower positive eigenvalue bounds using leading block augmentation

To illustrate the challenge posed by the problem in hand, recall the following result of Rusten and Winther [11, Lemma 2.1]. In their analysis it is assumed that A is positive definite (as opposed to semidefinite); however, the proof of this lemma does not rely on this, so the result still holds when A is semidefinite.

Lemma 2. *Let the eigenvalues of A lie in $[\mu_{\min}(A), \mu_{\max}(A)]$, with $\mu_{\min}(A) \ge 0$, and the singular values of B lie in $[\sigma_{\min}(B), \sigma_{\max}(B)]$, with $\sigma_{\min}(B) > 0$. Then the eigenvalues of $\mathcal{K}$ (2) are bounded in the union of intervals*

$$I^- \cup I^+,$$

where

$$I^- = \left[\frac{1}{2}(\mu_{\min}(A) - \sqrt{\mu^2_{\min}(A) + 4\sigma^2_{max}(B)}), \frac{1}{2}(\mu_{\max}(A) - \sqrt{\mu^2_{\max}(A) + 4\sigma^2_{\min}(B)})\right]$$

and

$$I^+ = \left[\mu_{\min}(A), \frac{1}{2}(\mu_{\max}(A) + \sqrt{\mu^2_{\max}(A) + 4\sigma^2_{\max}(B)})\right].$$

When A is singular, the upper bounds on both positive and negative values of $\mathcal{K}$ are unchanged, and the lower negative bound reduces to $-\sigma_{\max}(B)$. The main difficulty is that the lower bound on positive eigenvalues reduces to zero, which is not a useful bound, especially in situations where $\mathcal{K}$ is known to be nonsingular (which is our assumption throughout this paper). When the null spaces of A and B are well separated, the matrix $\mathcal{K}$ may in fact be well-conditioned and its minimal positive eigenvalue bounded away from zero.

As a motivating example that illustrates the range of possibilities, consider the coefficient matrix

$$\mathcal{K} = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 0 & b_2 \\ b_1 & b_2 & 0 \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}, \tag{3}$$

with $b_1^2 + b_2^2 = 1$ and $b_1, b_2 > 0$. The eigenvalues of A and singular value of B are the same for all such b_1, b_2, but the lowest positive eigenvalue of $\mathcal{K}$ varies depending on b_1 and b_2. The eigenvalues λ of $\mathcal{K}$ are the roots of the cubic polynomial $p(\lambda) = \lambda^3 - \lambda^2 - \lambda + b_2^2$. This polynomial has two positive roots and one negative root [17, Corollary 2.2]; the smaller positive root approaches zero as b_2 goes to zero (i.e., when A and B have overlapping null spaces), but as b_2 goes to 1 (i.e., when A and B have orthogonal null spaces) the smaller positive root approaches 1.

We now present a general approach for deriving nonzero bounds for the lower positive eigenvalues of $\mathcal{K}$ when A is singular. We recall the following result [22,23]:

Lemma 3. *Let*

$$\mathcal{K}(W) = \begin{bmatrix} A + B^T W B & B^T \\ B & 0 \end{bmatrix}, \tag{4}$$

where $W \in \mathbb{R}^{m\times m}$. If $\mathcal{K}$ and $\mathcal{K}(W)$ are both nonsingular, then

$$\mathcal{K}^{-1} = (\mathcal{K}(W))^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix}. \tag{5}$$

We will assume that W is positive semidefinite and the leading block $A_W := A + B^T W B$ of $\mathcal{K}(W)$ is positive definite. We can use this along with (5) to derive a nonzero bound on the lower positive eigenvalues of $\mathcal{K}$, using a free matrix parameter W.

Theorem 4. *Let $W \in \mathbb{R}^{m\times m}$ be a symmetric positive semidefinite matrix and let $A_W = A + B^T W B$ be positive definite. Then the positive eigenvalues of $\mathcal{K}$ are greater than or equal to*

$$\min \left\{ \mu_{\min}(A_W), \frac{1}{\mu_{\max}(W)} \right\}.$$

Proof. We derive a lower bound on the positive eigenvalues of $\mathcal{K}$ by considering an upper bound on the eigenvalues of $\mathcal{K}^{-1}$. By combining [2, Equation (3.4)] and (5), we obtain

$$\mathcal{K}^{-1} = \begin{bmatrix} A_W^{-1} - A_W^{-1} B^T S_W^{-1} B A_W^{-1} & A_W^{-1} B^T S_W^{-1} \\ S_W^{-1} B A_W^{-1} & -S_W^{-1} + W \end{bmatrix}, \tag{6}$$

where $S_W = B A_W^{-1} B^T$. Notice that we can write

$$\mathcal{K}^{-1} = \begin{bmatrix} A_W^{-1} & 0 \\ 0 & W \end{bmatrix} - \begin{bmatrix} A_W^{-1} B^T \\ -I \end{bmatrix} S_W^{-1} \begin{bmatrix} B A_W^{-1} & -I \end{bmatrix}.$$

Because the subtracted term is positive semidefinite, we conclude that the eigenvalues of $\mathcal{K}^{-1}$ are less than or equal to the eigenvalues of

$$\begin{bmatrix} A_W^{-1} & 0 \\ 0 & W \end{bmatrix}.$$

The stated result follows. □

3. Augmentation-based bounds when $W = \gamma I$

As in Section 2, we consider the augmented matrix $\mathcal{K}(W)$, but in this case we restrict ourselves to the case where

$$W = \gamma I,$$

as is done in [23,24]. For simplicity we write

$$A_\gamma = A + \gamma B^T B; \qquad \mathcal{K}_\gamma = \mathcal{K}(\gamma I).$$

In this case, the lower bound on positive eigenvalues presented in Theorem 4 reduces to $\min\left\{\mu_{\min}(A_\gamma), \frac{1}{\gamma}\right\}$.

We first consider the special case where $\mathrm{rank}(A) = n - m$ and $\mathcal{K}$ is nonsingular. We say here that A is *lowest-rank* because, if its rank were any lower, then $\mathcal{K}$ would necessarily be singular. It was shown in [24,25] that A_γ and $\mathcal{K}_\gamma$ have unique properties, which we will use here to refine the bound on lower positive eigenvalues given in Theorem 4. We return in Section 3.2 to the general case, where A is assumed to be rank-deficient but not lowest-rank.

3.1. Bounds when $\mathrm{rank}(A) = n - m$

Theorem 5. *When $\mathrm{rank}(A) = n - m$ and $A_\gamma = A + \gamma B^T B$, we have*

$$\mu_{\min}(A_\gamma) \geq \rho \cdot \min\left\{\mu^+_{\min}(A), \gamma\sigma^2_{\min}(B)\right\}, \tag{7}$$

where $\rho \leq 1$ is a constant that does not depend on γ.

Proof. We begin by writing a decomposition of A_γ as was done in [24]. Let

$$A = U\Lambda U^T, \quad B = QSV^T \tag{8}$$

be the reduced (economy-size) singular value decompositions of A and B.

The matrices $\Lambda \in \mathbb{R}^{(n-m)\times(n-m)}$ and $U \in \mathbb{R}^{n\times(n-m)}$ comprise the eigenpairs of A that correspond to its nonzero eigenvalues, and the columns of $V \in \mathbb{R}^{n\times m}$ are the set of eigenvectors of $B^T B$ that correspond to its nonzero eigenvalues. We can then write

$$A_\gamma = P\Sigma P^T, \tag{9}$$

where

$$P = \begin{bmatrix} U & V \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Lambda & 0 \\ 0 & \gamma S^2 \end{bmatrix}.$$

The decomposition in (9) resembles an eigenvalue decomposition, but is not an eigenvalue decomposition in general because the columns of V will not be orthogonal to those of U.

We then derive a lower bound on the eigenvalues of A_γ by obtaining an upper bound on the eigenvalues of A_γ^{-1}. We can write

$$\mu_{\max}(A_\gamma^{-1}) = \|A_\gamma^{-1}\| = \|P^{-T}\Sigma^{-1}P^{-1}\| \leq \|\Sigma^{-1}\| \cdot \|P^{-1}\|^2. \tag{10}$$

The largest eigenvalue of Σ^{-1} is equal to $\max\left\{\frac{1}{\mu^+_{\min}(A)}, \frac{1}{\gamma\sigma^2_{\min}(B)}\right\}$. The stated result follows by setting $\rho = \|P^{-1}\|^{-2}$ in (10). We claim that $\rho \leq 1$, with equality when U and V are mutually orthogonal (that is, when the range of A is orthogonal to the range of B^T). To show that this is the case, consider $x \in \ker(A)$. We then have

$$P^T x = \begin{bmatrix} U^T x \\ V^T x \end{bmatrix} = \begin{bmatrix} 0 \\ V^T x \end{bmatrix}.$$

Defining $q = P^T x$, since V is orthogonal we have

$$\|q\| \leq \|P^{-T}q\| \leq \|P^{-T}\|\|q\|,$$

meaning that $\|P^{-T}\|$ (and therefore $\|P^{-1}\|$) is greater than or equal to 1. Thus, $\rho \leq 1$. □

We now provide a value for $\rho = \|P^{-1}\|^{-2}$ in terms of the principal angles between $\mathrm{range}(A)$ and $\mathrm{range}(B^T)$. Let

$$\theta_i, i = 1, \ldots, r,$$

where $r = \min\{n - m, m\}$, denote these angles. The cosines $\cos(\theta_i)$ of these angles are given by the singular values of $U^T V$ (or $V^T U$).

Lemma 6. *When $\mathrm{rank}(A) = n - m$, let $\theta^{\min}$ denote the minimum principal angle between $\mathrm{range}(A)$ and $\mathrm{range}(B^T)$, and let $P = \begin{bmatrix} U & V \end{bmatrix} \in \mathbb{R}^{n\times n}$ where the columns of U are the eigenvectors corresponding to the nonzero eigenvalues of A and the columns of V are the right singular vectors of B. Then*

$$\|P^{-1}\| = \frac{1}{\sqrt{1 - \cos(\theta^{\min})}},$$

which implies that ρ defined in (7) is given by

$$\rho = 1 - \cos(\theta^{\min}).$$

Proof. We proceed by analyzing the eigenvalues of P^TP, using the fact that

$$\|P^{-1}\| = \frac{1}{\sqrt{\mu_{\min}(P^TP)}}.$$

We write P^TP in block form:

$$P^TP = \begin{bmatrix} U^T \\ V^T \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix} = \begin{bmatrix} I & U^TV \\ V^TU & I \end{bmatrix}.$$

The (1,1)-block of P^TP is size $(n-m)\times(n-m)$ and the (2,2)-block is size $m \times m$. We now assume without loss of generality that $n - m \geq m$. (If $n - m < m$, we can reorder the blocks of P^TP such that the (1,1)-block is larger, and use the same analysis as below.)

Letting $v = \begin{bmatrix} x^T & y^T \end{bmatrix}^T$ be an appropriately partitioned eigenvector, we write the eigenvalue equations for P^TP:

$$x + U^TVy = \lambda x; \tag{11a}$$

$$V^TUx + y = \lambda y. \tag{11b}$$

There is an eigenvalue $\lambda = 1$ with multiplicity $n - 2m$, which we observe by choosing $x \in \ker(V^TU)$ (because $V^TU \in \mathbb{R}^{(n-m)\times m}$ and has full rank if $\mathcal{K}$ is nonsingular, the dimension of the kernel is $n-2m$) and $y = 0$. For the remaining $2m$ eigenvalues, we assume $\lambda \neq 1$. From (11a) we have $x = \frac{1}{\lambda-1}U^TVy$, which we substitute into (11b) to obtain

$$y = \frac{1}{(\lambda-1)^2}V^TUU^TVy. \tag{12}$$

The eigenvalues of V^TUU^TV are given by $\cos^2(\theta_i)$, where θ_i are the principal angles between range(A) and range(B^T). Thus, for each θ_i we can write (12) as

$$y = \frac{\cos^2(\theta_i)}{(\lambda_i - 1)^2}y,$$

implying that

$$\lambda_i = 1 \pm \cos(\theta_i).$$

Thus each θ_i yields two distinct eigenvalues. Together with the $n - 2m$ eigenvalues with $\lambda = 1$, this accounts for all n eigenvalues of P^TP. Therefore, the smallest eigenvalue of P^TP is given by $1 - \cos(\theta^{\min})$; the stated result follows. □

We can use the results we have established for matrices with lowest-rank A to derive a lower bound on the positive eigenvalues of $\mathcal{K}$ that does not require us to know the eigenvalues of A_γ. We saw in Theorem 4 that for $W = \gamma I$, the bound is given by $\min\left\{\mu_{\min}(A_\gamma), \frac{1}{\gamma}\right\}$. As γ decreases, the value of $\mu_{\min}(A_\gamma)$ approaches zero (because A_γ approaches A); thus, we achieve the best possible lower bound when

$$\frac{1}{\gamma} = \mu_{\min}(A_\gamma).$$

Since we do not generally know the value of $\mu_{\min}(A_\gamma)$, we can instead select $\frac{1}{\gamma}$ to be equal to the reciprocal of the lower bound on $\mu_{\min}(A_\gamma)$ given by Theorem 5 and Lemma 6. That is, we find a γ that satisfies

$$\frac{1}{\gamma} = \left(1 - \cos(\theta^{\min})\right) \min\left\{\mu^+_{\min}(A), \gamma\sigma^2_{\min}(B)\right\}.$$

Depending on which of the arguments to the min function is smaller, we either have

$$\frac{1}{\gamma} = \mu^+_{\min}(A)(1 - \cos(\theta^{\min}))$$

or we have $\frac{1}{\gamma} = (1 - \cos(\theta^{\min})) \cdot \gamma\sigma^2_{\min}(B)$, which implies that

$$\frac{1}{\gamma} = \sigma_{\min}(B)\sqrt{1 - \cos(\theta^{\min})}.$$

Therefore, if we select

$$\frac{1}{\gamma} = \min\left\{\mu^+_{\min}(A)\left(1 - \cos(\theta^{\min})\right), \sigma_{\min}(B)\sqrt{1 - \cos(\theta^{\min})}\right\},$$

we know that $\mu_{\min}(A_\gamma)$ will be greater than or equal to this value of $\frac{1}{\gamma}$. This gives the following result:

Theorem 7. *Let* $\mathrm{rank}(A) = n - m$ *and let* $\theta^{\min}$ *denote the minimum principal angle between* $\mathrm{range}(A)$ *and* $\mathrm{range}(B^T)$*. The positive eigenvalues of* $\mathcal{K}$ *are greater than or equal to*

$$\min\left\{\mu^+_{\min}(A)\left(1-\cos(\theta^{\min})\right), \sigma_{\min}(B)\sqrt{1-\cos(\theta^{\min})}\right\}.$$

Remark 8. Depending on the properties of the problem, there may be no way to avoid a bound very close to zero. Specifically, if $\cos(\theta^{\min}) \lessapprox 1$, then the range of A and the range of B^T contain vectors that are nearly linearly dependent. When $\mathrm{rank}(A) = n - m$, this implies that we are close to the case where $\mathrm{rank}(\mathrm{range}(A) \cap \mathrm{range}(B^T)) < n$, which yields a singular $\mathcal{K}$. Thus, $\cos(\theta^{\min}) \lessapprox 1$ implies that $\mathcal{K}$ is ill-conditioned and a near-zero eigenvalue bound is appropriate.

Remark 9. The computation of $\cos(\theta^{\min})$ is somewhat involved, as it requires a full eigenvalue decomposition of A and a singular value decomposition of B. As such, it is mainly of theoretical value. That said, the quantity required is the *maximal* singular value of U^TV, which is computationally cheap once U and V are available. In contrast, the bound in [12] requires one or more of the *smallest* cosine values (and therefore the smallest singular values), which is more expensive to compute.

In some cases, more may be known about the null spaces of A and B than the ranges of A and B^T. For these settings, it is convenient to re-frame the result of Theorem 7 to rely on the angle between kernels rather than the angle between ranges. Because $\ker(A)$ and $\ker(B)$ are the respective orthogonal complements of $\mathrm{range}(A)$ and $\mathrm{range}(B^T)$, the principal angles are the same as those between $\ker(A)$ and $\ker(B)$. The following result then holds.

Corollary 10. *Let* $\mathrm{rank}(A) = n-m$ *and let* $\psi^{\min}$ *denote the minimum principal angle between* $\ker(A)$ *and* $\ker(B)$*. The positive eigenvalues of* $\mathcal{K}$ *are greater than or equal to*

$$\min\left\{\mu^+_{\min}(A)\left(1-\cos(\psi^{\min})\right), \sigma_{\min}(B)\sqrt{1-\cos(\psi^{\min})}\right\}.$$

3.2. *Bounds when* $\mathrm{rank}(A) \geq n - m$

We now return to the case in which A is rank-deficient but not lowest rank, and discuss how the results of the previous section can be extended to this case.

As before, if we consider a weight matrix $W = \gamma I$, a lower bound on the positive eigenvalues of $\mathcal{K}$ is given by

$$\min\left\{\frac{1}{\gamma}, \mu_{\min}(A_\gamma)\right\},$$

as this bound does not depend on the nullity of A. When A is not lowest rank, the bound of Theorem 5 for $\mu_{\min}(A_\gamma)$ is not immediately applicable.

We consider an additive splitting of A:

$$A = A_1 + A_2, \tag{13}$$

where $\mathrm{rank}(A_1) = n - m$ and A_1, A_2 are positive semidefinite. The eigenvalues of A_γ are all greater than or equal to those of

$$A_1 + \gamma B^T B =: \bar{A}_\gamma.$$

Let $\bar{\theta}^{\min}$ denote the minimum principal angle between $\mathrm{range}(A_1)$ and $\mathrm{range}(B^T)$. By Theorem 5 and Lemma 6, we have

$$\mu_{\min}(A_\gamma) \geq \mu_{\min}(\bar{A}_\gamma) \geq \left(1-\cos(\bar{\theta}^{\min})\right) \cdot \min\left\{\mu^+_{\min}(A_1), \gamma\sigma^2_{\min}(B)\right\}.$$

As we did before, we can select $\frac{1}{\gamma}$ to be equal to the smaller of these two values to obtain a lower bound on the positive eigenvalues of $\mathcal{K}$ that does not require forming an augmented matrix. The proof of the following theorem is similar to that of Theorem 7 and is omitted.

Theorem 11. *Let* A_1, A_2 *be arbitrary positive semidefinite matrices satisfying* $A_1+A_2 = A$ *and* $\mathrm{rank}(A_1) = n-m$*. The positive eigenvalues of* $\mathcal{K}$ *are greater than or equal to*

$$\min\left\{\mu^+_{\min}(A_1)\left(1-\cos(\bar{\theta}^{\min})\right), \sigma_{\min}(B)\sqrt{1-\cos(\bar{\theta}^{\min})}\right\},$$

where $\bar{\theta}^{\min}$ *denotes the minimum principal angle between* $\mathrm{range}(A_1)$ *and* $\mathrm{range}(B^T)$*.*

Obviously, the correct choice of splitting to obtain the rank-$(n-m)$ matrix A_1 is non-trivial. The choice that maximizes the value $\mu^+_{\min}(A_1)$ is to take $A_1 = U^{\max}_{n-m}\Lambda^{\max}_{n-m}(U^{\max}_{n-m})^T$, where $\Lambda^{\max}_{n-m} \in \mathbb{R}^{(n-m)\times(n-m)}$ is a diagonal matrix of the $n-m$ largest eigenvalues of A and $U^{\max}_{n-m} \in \mathbb{R}^{n\times(n-m)}$ is a matrix of the corresponding eigenvectors. The resulting bound for this choice of A_1 is presented below as a corollary to Theorem 11.

Corollary 12. *Let A be semidefinite with $n-m \le \operatorname{rank}(A) \le n$. The positive eigenvalues of $\mathcal{K}$ are greater than or equal to*

$$\min\left\{\mu_{n-m+1}(A)\left(1-\cos(\theta_{n-m}^{\min})\right), \sigma_{\min}(B)\sqrt{1-\cos(\theta_{n-m}^{\min})}\right\},$$

where $\mu_{n-m+1}(A)$ denotes the $(n-m+1)$th largest eigenvalue of A and $\theta_{n-m}^{\min}$ the smallest principal angle between $\operatorname{range}(B^T)$ and the subspace spanned by the eigenvectors corresponding to the $n-m$ largest eigenvalues of A. (Or, equivalently, $\theta_{n-m}^{\min}$ is the smallest principal angle between $\ker(B)$ and the subspace spanned by the eigenvectors corresponding to the m smallest eigenvalues of A – see Corollary 10.)

We note that the choice of A_1 in Corollary 12 is not always ideal, in that it may lead to an overly pessimistic bound. For example, consider the matrix (with $n=3$ and $m=2$):

$$\mathcal{K} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] =: \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}, \tag{14}$$

where $0 < \alpha < 1$. The positive eigenvalues of $\mathcal{K}$ are α, 1, and $\frac{1+\sqrt{5}}{2}$. The eigenvector $U_{n-m}^{\max}$ that we retain to form A_1, which is in this case the eigenvector corresponding to $\lambda = 1$, is:

$$U_{n-m}^{\max} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Because this eigenvector is in the range of B^T, the value $\theta_{n-m}^{\min}$ is 0, meaning that Corollary 12 gives a bound of 0. We would obtain a better bound if, instead of keeping the part of the spectrum of A that corresponds to the eigenvalue $\lambda = 1$, we kept the portion of the spectrum corresponding to $\lambda = \alpha$ (this would in fact give a tight bound of α).

An alternative option is to select A_1 in a way that attempts to minimize the value $\cos(\bar{\theta}_{\min})$. Even restricting ourselves to A_1 of the form $A_1 = U_{sub}\Lambda_{sub}U_{sub}^T$, where Λ_{sub} and U_{sub} respectively contain some subset of $n-m$ nonzero eigenvalues of A and the corresponding eigenvectors, there are combinatorially many options to consider, making a brute force approach impractical.

We describe a simple greedy strategy below. Let v_j represent the jth column of the matrix V of right singular vectors of B, as defined in (8). Because we want to select a matrix U_{sub} that minimizes the maximum singular value of $U_{sub}^T V$, we select the $(n-m)$ eigenvectors u_i corresponding to positive eigenvalues for which the values of $\max_{j=1,\ldots,m} |u_i^T v_j|$ are the smallest. If we use this strategy on the matrix $\mathcal{K}$ in (14), the matrix A_1 will consist of the spectrum of A corresponding to the eigenvalue α, and Theorem 11 will yield a tight bound of α.

3.3. Comparison to other bounds

Here we include a comparison between our eigenvalue bounds and those of other sources. As mentioned in the introduction, the bound of Rusten and Winther [11] reduces to zero in the case we consider here with singular A.

We now include a more detailed comparison of our bound and that of Ruiz et al. [12, Theorem 4.1], stated below:

Theorem 13. *Consider a saddle-point matrix $\mathcal{K}$ where A satisfies $\mu_{\max} = 1$ and B satisfies $BB^T = I$ (i.e., the columns of B are orthonormal). Let $\theta_p^{\max}$ denote the maximum principal angle between $\operatorname{range}(U_p^{\min})$ and $\operatorname{range}(B^T)$, where the columns of $U_p^{\min}$ are the eigenvectors corresponding to the p smallest eigenvalues of A. Then the positive eigenvalues of $\mathcal{K}$ are greater than or equal to*

$$\max_{1\le p\le r-1} \min\left(\frac{\mu_{n-p+1}(A)}{2}, \frac{\mu_{\min}(A) + \frac{4}{5}\cos^2(\theta_p^{\max})\mu_{n-p+1}(A)}{1+\frac{4}{5}\cos^2(\theta_p^{\max})\mu_{n-p+1}(A)}\right),$$

where $r = \min(m, n-m)$.

We note that our bound is applicable to a wider class of matrices because we do not require that the largest eigenvalue of A be equal to 1 or that the columns of B be orthonormal. The former condition is easily addressed by scaling the matrix $\mathcal{K}$, which still allows us to easily recover the eigenvalues of the original matrix. However, there is no way to orthonormalize the columns of B without changing the eigenvalues of $\mathcal{K}$ in a way that does not allow us to easily map the eigenvalues of the modified system to those of the original system; thus, Theorem 13 is less applicable for general B.

A consequence of Theorem 13 is that the lower positive eigenvalue bound can be no greater than the rth smallest eigenvalue of A, where $r = \min(m, n-m)$. Thus, this bound reduces to zero for matrices where A has sufficiently high nullity, including when A is lowest-rank.

For matrices with orthonormal B and the nullity of A less than r, sometimes our bound is tighter and sometimes the bound of Ruiz et al. is tighter. In particular, the quality of our bound where A is not lowest-rank is vulnerable to the choice of matrix A_1 in Theorem 11 and/or Corollary 12. As an example where Corollary 12 gives a tighter bound, consider the following example with $n = 4$ and $m = 2$:

$$\mathcal{K} = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 1 \\ 0 & 0 & 2\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] =: \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}, \tag{15}$$

where $0 < \alpha < 0.5$. The positive eigenvalues of $\mathcal{K}$ are given in ascending order by 2α, 1 (with multiplicity 2), and $\frac{\alpha+\sqrt{\alpha^2+4}}{2}$. For our bound, the matrix $U_{n-m}^{\max}$ corresponding to the largest eigenvectors of A is given by

$$U_{n-m}^{\max} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Because this is orthogonal to range(B^T), the value $\cos(\theta_{n-m}^{\min})$ is equal to 0, and the bound of Corollary 12 is equal to α.

For the bound of Ruiz et al. we only need to consider $p = 1$ (because $r = 2$). The vector corresponding to the smallest eigenvector of A is defined by

$$U_p^{\min} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which is in range(B^T), and we therefore have $\theta_p^{\max} = 0$. The bound of Theorem 13 reduces to

$$\min\left(\frac{\alpha}{2}, \frac{0 + \frac{4\alpha}{5}}{1 + \frac{4\alpha}{5}}\right) \le \frac{\alpha}{2} < \alpha,$$

and is thus looser than the bound given by Corollary 12.

If, on the other hand, we modify $\mathcal{K}$ in (15) to have

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the smallest positive eigenvalue of $\mathcal{K}$ will be 2α, the bound of Ruiz et al. will still be $\min\left(\frac{\alpha}{2}, \frac{0+\frac{4\alpha}{5}}{1+\frac{4\alpha}{5}}\right)$, while Corollary 12 will give a bound of zero (though this can be improved by selecting a different matrix A_1 and instead using Theorem 11 – for instance, greedily retaining the eigenvectors u_i that minimize $\max_{i=1,\ldots,j} |u_i^T v_j|$, as described at the end of Section 3.2, yields a bound of α).

3.4. Preconditioning

We will briefly discuss how our eigenvalue bounds could be used to derive and analyze effective preconditioning strategies, in the spirit of the useful discussion in Ruiz et al. [12, Section 4]. Let us consider a general positive definite block diagonal preconditioner

$$\mathcal{M} = \begin{bmatrix} M & 0 \\ 0 & S \end{bmatrix},$$

where $M \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{m\times m}$ are positive definite. Then the split preconditioned operator

$$\mathcal{M}^{-1/2}\mathcal{K}\mathcal{M}^{-1/2} = \begin{bmatrix} M^{-1/2}AM^{-1/2} & M^{-1/2}B^TS^{-1/2} \\ S^{-1/2}BM^{-1/2} & 0 \end{bmatrix}$$

is also a saddle-point matrix with a singular leading block. In order to obtain better performance of an iterative solver such as MINRES, we want to move the smallest positive eigenvalues of $\mathcal{M}^{-1/2}\mathcal{K}\mathcal{M}^{-1/2}$ away from zero (and, ideally, close to the largest positive eigenvalues so that all the positive eigenvalues are contained in a small interval). Theorem 7 tells us that there are three ways in which we can do this:

(i) increase the smallest positive eigenvalue of $M^{-1/2}AM^{-1/2}$;
(ii) increase the smallest singular value of $S^{-1/2}BM^{-1/2}$;
(iii) increase the minimum principal angle between range($M^{-1/2}AM^{-1/2}$) and range($S^{-1/2}BM^{-1/2}$).

In the lowest-rank case, the best case in terms of condition (iii) is for range($M^{-1/2}AM^{-1/2}$) and range($S^{-1/2}BM^{-1/2}$) to be mutually orthogonal. See [1], [26, Lemma 5], for an example of how this applies to a time-harmonic Maxwell model problem where A is lowest-rank.

4. Numerical experiments

We test our eigenvalue bounds on two problems. The first is an electromagnetics model problem described in [1]. Consider the time-harmonic Maxwell equations in lossless media with perfectly conducting boundaries and constant coefficients. The problem is to find the vector field u and multiplier p such that

$$\begin{aligned} \nabla \times \nabla \times u + \nabla p &= f \text{ in } \Omega, \\ \nabla \cdot u &= 0 \text{ in } \Omega, \\ u \times n &= 0 \text{ on } \partial\Omega, \\ p &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Discretizing with Nédéléc finite elements for u and nodal elements for p [27] yields a linear system of the form

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix},$$

where A is a discrete curl–curl operator and B is a discrete divergence operator.

In the above-described problem, A has rank $n - m$, and hence it is lowest rank per the terminology we use in this paper. Fig. 1 shows the predicted bound (as a solid line), the actual smallest positive eigenvalue (dashed line) for various values of γ for a Maxwell matrix on the domain $\Omega = [0, 1] \times [0, 1]$ with $n = 6,080$ and $m = 1,985$.

The second problem describes linear systems arising from an interior point method (IPM) solution to a quadratic program (QP); see [28] and the references therein for a detailed description. At each iteration of the IPM, we solve a linear system with a matrix of the form

$$\mathcal{K} = \begin{bmatrix} H + X^{-1}Z & J^T \\ J & 0 \end{bmatrix},$$

where H and J are respectively the Hessian and Jacobian matrices for the QP, and X and Z are diagonal matrices of the current primal and dual iterates, some entries of which go to 0 as the iterations progress. Thus, the leading block becomes progressively more ill-conditioned as the iterations proceed.

In Fig. 2 we show the results of the bound of Corollary 12 on the first IPM iteration on TOMLAB[1] Problem 17 for which the saddle-point matrix $\mathcal{K}$ is numerically singular. This problem has $n = 293$ and $m = 286$. For the particular matrix shown in the experiment below (which arises in the 12th iteration of the IPM algorithm of [29]), there are 115 "numerically zero" eigenvalues of the leading block (which we define as those less than machine epsilon times the largest eigenvalue of that block).

In both cases the actual smallest positive eigenvalue $\mu_{\min}(\mathcal{K})$ occurs precisely where $\frac{1}{\gamma} = \mu_{\min}(A_\gamma)$. The bounds for the Maxwell matrix are rather tight, in the sense that they are of the same order of magnitude as the eigenvalue (we also see this with Maxwell matrices of other sizes): the predicted eigenvalue bound is 0.0453 while the actual smallest positive eigenvalue is 0.0611.

The bound for the TOMLAB problem is looser: the predicted bound is 4.716×10^{-7} while the actual smallest positive eigenvalue is 1.817×10^{-4}. Recall that our approach for deriving the bound for a matrix with A that does not have the lowest rank consisted of two steps: (1) implicitly convert the matrix to one with a lowest-rank leading block (denoted by A_1 in Theorem 11) by subtracting some positive semidefinite portion of A; and (2) estimate the lower bound for the matrix with the lowest-rank leading block using the results of Section 3.1, using the fact that this will also be a lower bound for the original matrix. Because our analysis relies on "dropping" a positive semidefinite part of A we might in general expect that to lead to some looseness in the bound.

However, the dropping is not the cause of the looseness in this case of the TOMLAB problem, as the saddle-point matrix we obtain by simply replacing A with the dropped portion A_1 that is used in Corollary 12 has almost the same smallest positive eigenvalue as the original matrix (1.810×10^{-4}, compared with 1.817×10^{-4}). Thus, the looseness in this bound does not come from the dropping part of the spectrum of A to create a lowest-rank matrix, but rather in the estimation of the lower positive eigenvalue bound of the modified matrix.

[1] Test matrices available at https://tomopt.com/tomlab/.

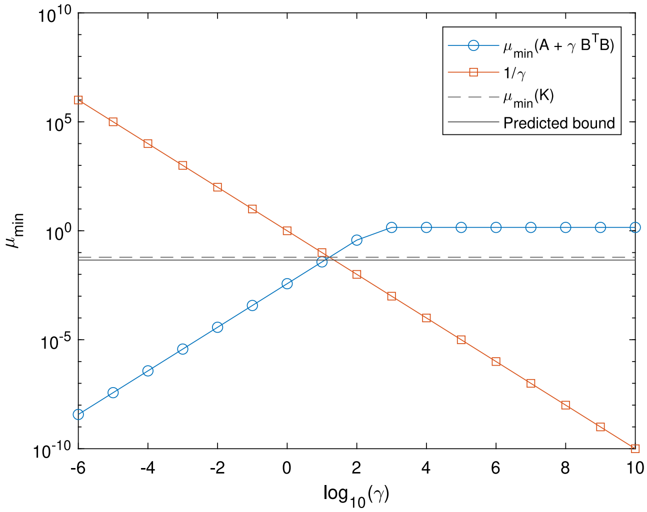

Fig. 1. Comparison of predicted and actual smallest positive eigenvalue bounds at various values of γ for the Maxwell matrix (lowest rank).

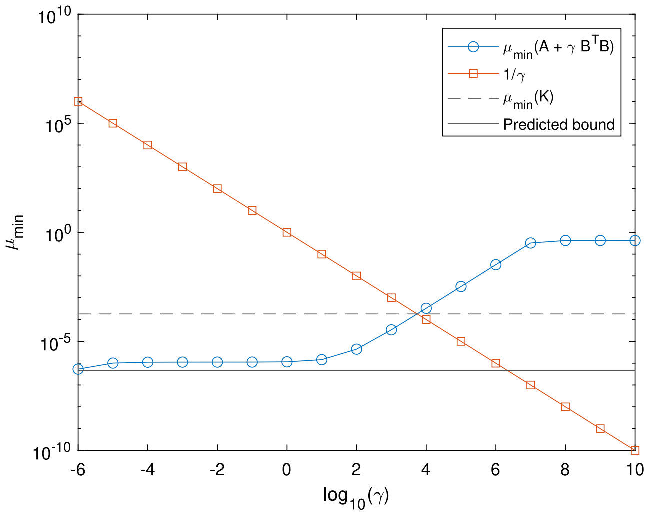

Fig. 2. Comparison of predicted and actual smallest positive eigenvalue bounds at various values of γ for the IPM matrix for TOMLAB QP 17.

5. Conclusions

We have described a novel framework for bounding eigenvalues of saddle-point matrices by strategically augmenting some of their blocks. We used this approach to derive (nonzero) bounds on the lower positive eigenvalues of saddle-point matrices with singular leading blocks. By making certain assumptions on the augmentation parameters, we were able to derive an eigenvalue bound that does not require the formation of an augmented matrix.

Future work may include improving the bound in the non-lowest-rank case (for instance, by judiciously selecting the portion of the spectrum of A that is "dropped") and using this framework to analyze the convergence of preconditioned iterative solvers. An understanding of how the spectral properties of saddle-point matrices depend on the interactions between the blocks is also useful in developing preconditioning approaches, and this is the subject of a follow-up paper [26].

Data availability

Data will be made available on request.

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