# Table Cartograms

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**Abstract.** A table cartogram of a two dimensional $m \times n$ table $A$ of non-negative weights in a rectangle $R$, whose area equals the sum of the weights, is a partition of $R$ into convex quadrilateral faces corresponding to the cells of $A$ such that each face has the same adjacency as its corresponding cell and has area equal to the cell's weight. Such a partition acts as a natural way to visualize table data arising in various fields of research. In this paper, we give a $O(mn)$-time algorithm to find a table cartogram in a rectangle. We then generalize our algorithm to obtain table cartograms inside arbitrary convex quadrangles, circles, and finally, on the surface of cylinders and spheres.

## 1 Introduction

A *cartogram*, or *value-by-area diagram*, is a thematic cartographic visualization, in which the areas of countries are modified in order to represent a given set of values, such as population, gross-domestic product, or other geo-referenced statistical data. Red-and-blue population cartograms of the United States were often used to illustrate the results in the 2000 and 2004 presidential elections. While geographically accurate maps seemed to show an overwhelming victory for George W. Bush, population cartograms effectively communicated the near 50-50 split, by deflating the rural and suburban central states.

The challenge in creating a good cartogram is thus to shrink or grow the regions in a map so that they faithfully reflect the set of pre-specified area values, while still retaining their characteristic shapes, relative positions, and adjacencies as much as possible. In this paper we introduce a new *table cartogram* model, where the input is a two dimensional $m \times n$ table of non-negative weights, and the output is a rectangle with area equal to the sum of the input weights partitioned into $m \times n$ convex quadrilateral faces each with area equal to the corresponding input weight. Fig. 1 shows two such examples. Such a visualization preserves both area and adjacencies, furthermore, it is simple, visually attractive, and applicable to many fields that require visualization of data table.

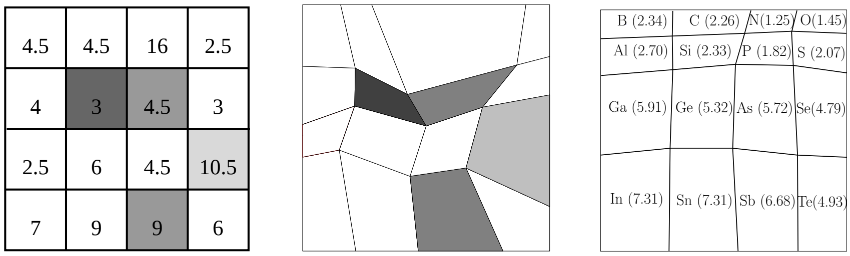


**Fig. 1.** A $4 \times 4$ table, its table cartogram, and a cartogram of some elements of the periodic table according to their density in grams per cubic centimeter (for solids) or per liter (for gases)

The solution to the problem is not obvious even for a $2 \times 2$ table. For example, Fig. 2(a–b) shows a table $A$ and a unit square $R$. One attempt to find the cartogram of $A$ in $R$ may be to first split $R$ horizontally according to the sum of each row, and then to find a good split in each subrectangle to realize the correct areas. But this approach does not work, because the first split prevents the creation of the two convex quadrilaterals with area $\epsilon$ in opposite corners that share a boundary vertex, Fig. 2(c). Fig. 2(d) shows a possible cartogram.

The following little argument shows that $2 \times 2$ table cartograms exist. The argument contains some elements that will be reused for the general case. The input is a $2 \times 2$ table with four positive reals $a, b, c, d$ with $a + b + c + d = 1$, as shown in Fig. 2(e). Rotational symmetry of the problem allows us to assume that $a + b \leq 1/2$. Fix the unit square $R$ with corners $(0,0), (0,1), (1,1), (1,0)$ as the frame for the table cartogram. Now consider the horizontal line $\ell$ with the property that every triangle $T(p)$ with top side equal to the top side of $R$ and one corner $p$ on $\ell$ has area $a + b$. Since $a + b \leq 1/2$, the line $\ell$ intersects $R$ in a horizontal segment. For $p \in \ell \cap R$, the vertical line through $p$ partitions $R \setminus T(p)$ into a left 4-gon $S^-$ and a right 4-gon $S^+$. The areas of these two 4-gons depend continuously on the position of point $p$ but their sum is always $c + d$. If $p$ is on the left boundary, $\mathit{Area}\,(S^+) = c + d$, and if $p$ is on the right boundary, $\mathit{Area}\,(S^+) = 0$. Hence, it follows from the intermediate value theorem that there is a position for $p$ on $\ell \cap R$ such that $\mathit{Area}\,(S^-) = c$ and $\mathit{Area}\,(S^+) = d$. By rotating a line around this $p$, we find a line that partitions $T(p)$ such that the

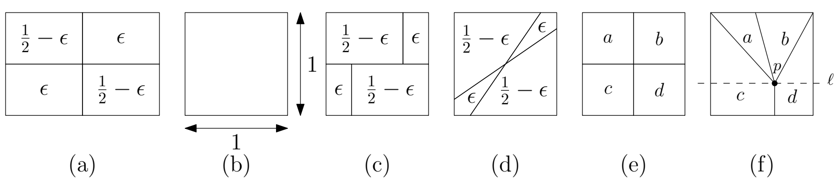


**Fig. 2.** (a) A $2 \times 2$ table $A$. (b) $R$. (c) An attempt to find a cartogram. (d) A cartogram of $A$ in $R$. (e) A $2 \times 2$ table $A$. (f) The cartogram showing $\ell$ as a dashed line.

left triangle has area $a$ and thus the right triangle has area $b$, this again uses the intermediate value theorem. The resulting partition of $R$ into four parts is a table cartogram for the input table, see Fig. 2(f). The critical reader may object that two of the 4-gons have a degenerate side. This can be avoided by perturbing the cartogram slightly to make a very short edge instead of a point. The result is an $\varepsilon$ approximate cartogram without degeneracies. Another approach is to modify the construction rules so that degeneracies are avoided. We take this approach in Section 2 to show the existence of non-degenerate table cartograms in general.

***Related Work.*** The problem of representing additional information on top of a geographic map dates back to the 19th century, and highly schematized rectangular cartograms can be found in the 1934 work of Raisz [16]. Recently, van Kreveld and Speckmann describe automated methods to produce rectangular cartograms [19]. With such rectangular cartograms it is not always possible to represent all adjacencies and areas accurately [12, 19]. However, in many “simple” cases, such as France, Italy and the USA, rectangular cartograms and even table cartograms offer a practical and straightforward schematization, e.g., Fig. 3. *Grid maps* are a special case of single-level spatial treemaps: the input is a geographic map mapped onto a grid of equal-sized rectangles, in such a way as to preserve as well as possible the relative positions of the corresponding regions [20, 9]. As we show, such maps can always be visualized as table cartograms.

Eppstein *et al.* studied area-universal rectangular layouts and characterized the class of rectangular layouts for which all area-assignments can be achieved with combinatorially equivalent layouts [8]. If the requirement that rectangles are used is relaxed to allow the use of rectilinear regions then de Berg *et al.* [4] showed that all adjacencies can be preserved and all areas can be realized with 40-sided regions. In a series of papers the polygon complexity that is sufficient to realize any rectilinear cartogram was decreased from 40 sides down to 8 sides [2], which is best possible due to an earlier lower bound [21].

More general cartograms without restrictions to rectangular or rectilinear shapes have also been studied. For example adjacencies can be preserved and areas represented perfectly using convex quadrilaterals if the dual of the map is

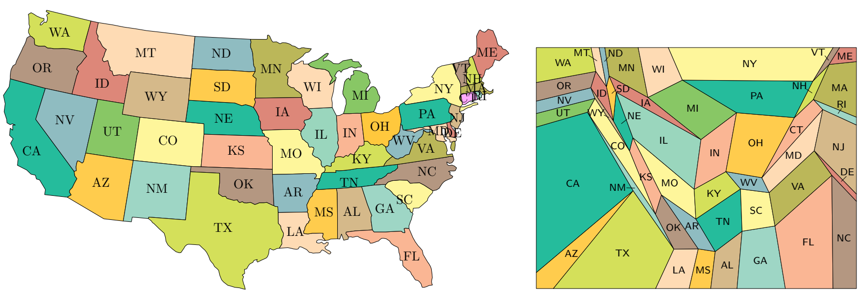


**Fig. 3.** A table cartogram of USA according to the population of the states in 2010, using the grid map of [9].

an outerplanar graph [1]. Dougenik *et al.* introduced a method based on force fields where the map is divided into cells and every cell has a force related to its data value which affects the other cells [6]. Dorling used a cellular automaton approach, where regions exchange cells until an equilibrium has been achieved, i.e., each region has attained the desired number of cells [5]. This technique can result in significant distortions, thereby reducing readability and recognizability. Keim *et al.* defined a distance between the original map and the cartogram with a metric based on Fourier transforms, and then used a scan-line algorithm to reposition the edges so as to optimize the metric [14]. Gastner and Newman [11] project the original map onto a distorted grid, calculated so that cell areas match the pre-defined values. The desired areas are then achieved via an iterative diffusion process inspired by physical intuition. The cartograms produced this way are mostly readable but the complexity of the polygons can increase significantly. Edelsbrunner and Waupotitsch [7] generated cartograms using a sequence of homeomorphic deformations. Kocmoud and House [13] described a technique that combines the cell-based approach of Dorling [5] with the homeomorphic deformations of Edelsbrunner and Waupotitsch [7].

There are thousands of papers, spanning over a century, and covering various aspects of cartograms, from geography to geometry and from interactive visualization to graph theory and topology. The above brief review is woefully incomplete; the survey by Tobler [18] provides a more comprehensive overview.

***Our Results.*** The main construction is presented in Section 2. We start with a simple constructive algorithm that realizes any table inside a rectangle in which each cell is represented by a convex quadrilateral with its prescribed weight. The approach relies on making many of the regions be triangles. We then modify the method to remove such degeneracies. The construction can be implemented to run in $O(mn)$ time, i.e., in time linear in the input size.

In Section 3 we find table cartograms inside arbitrary triangles or convex quadrilaterals, which is best possible, because regular $n$-gons, $n \geq 5$, do not always support table cartograms (e.g., consider a table with some cell value larger than the maximum-area convex quadrangle that can be drawn inside the $n$-gon). We also realize table cartograms inside circles, using circular-arcs, and on the surface of a sphere via a transformation from a realization on the cylinder.

## 2 Table cartograms in rectangles

We first construct a cartogram with degenerate 4-gons. The input is a table $A$ with $m$ rows and $n$ columns of non-negative numbers $A_{i,j}$. Let $S = \sum_{i,j} A_{i,j}$ and let $S_i$ be the sum of the numbers in row $i$, i.e., $S_i = \sum_{1\leq j\leq n} A_{i,j}$. Assume, by scaling, that $S > 4$. Let $R$ be the rectangle with corners $(0,0)$, $(S/2,0)$, $(S/2,2)$, $(0,2)$. We construct the cartogram within $R$ and later generalize the construction to all rectangles with area $S$. Let $k$ be the largest index such that the sum of the numbers in rows $1, 2, \ldots, k-1$ is less than $S/2$. We may then choose $\lambda \in (0,1]$ such that $\sum_{1\leq i\leq k-1} S_i + \lambda S_k = S/2$. We split the table $A$ into

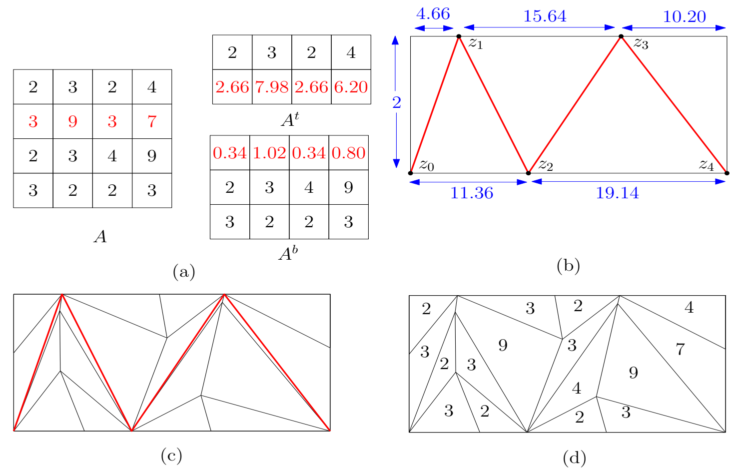


**Fig. 4.** (a) Illustration for $A, A^t$ and $A^b$, where $k = 2$ and $\lambda \approx 0.886$. (b) The zigzag path $Z$. We have distorted the aspect ratio of the figure to increase readability. (c) The subdivision of triangles, where $Z$ is shown in red, and (d) the complete cartogram.

two tables $A^t$ and $A^b$. Table $A^t$ consists of $k$ rows and $n$ columns. The first $k-1$ rows are taken from $A$, i.e., $A^t_{i,j} = A_{i,j}$ for $1 \le i \le k-1$ and $1 \le j \le n$. The last row is a $\lambda$-fraction of row $k$ from $A$, i.e., $A^t_{k,j} = \lambda \cdot A_{k,j}$ for all $j$. Table $A^b$ consists of $m-k+1$ rows and $n$ columns. The first row accommodates the remaining portion of row $k$ from $A$, i.e., $A^b_{1,j} = (1-\lambda) \cdot A_{k,j}$. All the other rows are taken from $A$, i.e., $A^b_{i,j} = A_{i+k-1,j}$ for $i > 1$ and all $j$. An example is shown in Fig. 4(a). If $\lambda = 1$, then $A^b$ contains a top row of zeros.

Let $D^t_j$ be the sum of entries in columns $2j-2$ and $2j-1$ from $A^t$, where $1 \le j \le \lfloor m/2+1 \rfloor$. Note that $D^t_1$ is only responsible for one column. The same may hold for the last $D^t_j$ depending on the parity of $m$. Similarly, $D^b_l$ is the sum of entries in columns $2l-1$ and $2l$ from $A^b$, where $1 \le l \le \lceil m/2 \rceil$. Again, depending on the parity of $m$ the last $D^b_l$ may only be responsible for one column.

We now define a zig-zag $Z$ in $R$ (formally, $Z$ is a polygonal line) such that the areas of the triangles defined by $Z$ are the numbers $D^t_1, D^b_1, D^t_2, D^b_2, D^t_3, \dots$ in this order. The zig-zag starts at $z_0 = (0,0)$. Since the height of $R$ is 2, the first segment ends at $z_1 = (D^t_1, 2)$ and the second segment goes down to $z_2 = (D^b_1, 0)$. In general, for $i$ odd, $z_i = (\sum_{j=1}^{\lceil i/2 \rceil} D^t_j, 2)$ and for $i$ even, $z_i = (\sum_{l=1}^{i/2} D^b_l, 0)$. An important property of $Z$ is that it ends at one of the two corners on the right side of $R$. This is because $\sum_j D^t_j = S/2 = \sum_l D^b_l$.

Lemma 2 shows that we can partition each triangle created by the zig-zag $Z$ into triangles whose areas are the corresponding entries in $A^t$ or $A^b$. It relies on the following lemma which is a consequence of properties of barycentric coordinates. We omit the proof due to space constraints.

**Lemma 1 (Triangle Lemma).** *Let $\triangle abc$ be a triangle and let $\alpha$, $\beta$, $\gamma$ be non-negative numbers, where $\alpha + \beta + \gamma = Area(\triangle abc)$. Then we can find a point $p$ in $\triangle abc$, where $Area(\triangle pbc) = \alpha$, $Area(\triangle apc) = \beta$, $Area(\triangle abp) = \gamma$, in $O(1)$ arithmetic operations.*

**Lemma 2.** *Let $A$ be an $m \times 2$ table such that each cell is assigned a non-negative number. Let $\triangle abc$ be a triangle such that the area of $\triangle abc$ is equal to the sum of the numbers of $A$. Then $A$ admits a cartogram inside $\triangle abc$ such that all cells of $A$ are represented by triangles and the boundary between those triangles representing cells in the left column and those representing cells in the right column is a polygonal path connecting point $a$ to some point on the segment $bc$.*

*Proof.* The proof is by induction on $m$. The case $m = 1$ is obvious. If $m > 1$ we define $\alpha = \sum_{1 \le i \le m-1} A_{i,1} + A_{i,2}$, $\beta = A_{m,1}$ and $\gamma = A_{m,2}$. Using Lemma 1 we find a point $p$ in $\triangle abc$ that partitions the triangle into triangles of areas $\alpha$, $\beta$ and $\gamma$. We keep the triangles $\triangle apc$ and $\triangle abp$ as representatives for $A_{m,1}$ and $A_{m,2}$ and construct the cartogram for the first $m-1$ rows of $A$ in the triangle $\triangle pbc$ by induction. □

To partition triangle $\triangle z_{2j-2}, z_{2j-3}, z_{2j-1}$, for $1 \le j \le \lfloor m/2 + 1 \rfloor$ (where $z_{-1} = (0, 2)$ and $z_{m+1} = (S/2, 2)$ if needed), we appeal to Lemma 2 with $A$ (in the lemma) being the two columns from $A^t$ whose sum is $D^t_j$. To make Lemma 2 applicable to cases like $D^t_1$ which represent only one column from $A^t$, we simply add a column of zeros to $A$. Similarly, we can partition triangle $\triangle z_{2l-1}, z_{2l-2}, z_{2l}$, for $1 \le l \le \lceil m/2 \rceil$.

This yields a table cartogram of the $(m+1) \times n$ table $A^+$ that is obtained by stacking $A^t$ on $A^b$. Note, however, that all triangles representing cells from the last row of $A^t$ have a side that equals one of the edges of $Z$. Symmetrically, all triangles representing cells from the first row of $A^b$ have a side on $Z$. Hence, by removing the edge of $Z$ we glue two triangles of area $\lambda A_{k,j}$ and $(1-\lambda)A_{k,j}$ into a 4-gon of area $A_{k,j}$. The 4-gons obtained by removing edges of $Z$ are convex because they have crossing diagonals. This completes the construction.

To complete the proof of the following theorem, in which $R$ is a $w \times h$ rectangle with area $S$, we scale the above cartogram by a factor of $h/2$ vertically and a factor of $2/h$ horizontally.

**Theorem 1.** *Let $A$ be an $m \times n$ table of non-negative numbers $A_{i,j}$. Let $R$ be a rectangle with width $w$, height $h$ and area equal to the sum of the numbers of $A$. Then there exists a cartogram of $A$ in $R$ such that every face in the cartogram is convex. The construction requires $O(mn)$ arithmetic operations.*

***Removing degeneracies.*** The construction of the proof of Theorem 1 creates faces of degenerate shape, i.e., some faces may not be perfect quadrangles, as shown in Fig. 4(d). We modify this construction to avoid the degeneracies. Of course we have to make a stronger assumption on the input: All entries $A_{i,j}$ of the table are strictly positive. The first part of the construction remains unaltered.

– Determine $k$ and $\lambda$ such that $\sum_{1 \le i \le k-1} S_i + \lambda S_k = S/2$.

– Define $A^t$ and $A^b$ and the two-column sums $D_j^t$ and $D_l^b$ for these tables.
– Compute the zig-zag in the rectangle $R$ of height 2 and width $S/2$.

Let $z_0, z_1, \ldots, z_n$ be the corner points of the zig-zag $Z$. For $i$ even we define $z'_i = z_i + (0, v)$ and for $i$ odd $z'_i = z_i - (0, v)$, i.e., $z'_i$ is obtained by shifting $z_i$ vertically a distance of $v$ into $R$. We will choose this positive value $v$ to obey conditions (B1) and (B2) required by the construction (these conditions have been specified later). Let $Z'$ be the zig-zag with corners $z'_0, z'_1, \ldots, z'_n$. The segment $z'_i, z_i$ is the *leg at* $z'_i$. The union of all the legs and $Z'$ is the *skeleton* $G'$ of a partition of $R$ into 5-gons. We refer to the 5-gons with corners $z_{i-1}, z_{i+1}, z'_{i+1}, z'_i, z'_{i-1}$ as $F_i$. We abstain from introducing extra notation for the two 4-gons at the ends of $Z'$ and just think of them as degenerate 5-gons.

**Lemma 3.** *A 5-gon in $R$ with vertices $(x_1, 0), (x_3, 0), (x_3, v), (x_2, 2 - v), (x_1, v)$ has the same area $x_3 - x_1$ as the triangle with corners $(x_1, 0), (x_3, 0), (x_2, 2)$.*

*Proof.* First note that changing the value of $x_2$ (shear) preserves the area of the 5-gon and of the triangle. Hence we may assume that $x_2 = x_3$. Now let $P$ be the parallelogram with corners $(x_1, 0), (x_1, v), (x_2, 2), (x_2, 2-v)$. Both, the 5-gon and the triangle can be partitioned into the triangle $(x_1, 0), (x_2, 2 - v), (x_3, 0)$ and a triangle that makes a half of $P$. $\square$

Some of the 5-gons $F_i$ may not be convex. However, concave corners can only be at $z'_{i+1}$ or $z'_{i-1}$. To get rid of concave corners we deal with corners at $z'_1, z'_2, \ldots, z'_{n-1}$ in this order. At each $z'_i$ we may slightly shift $z'_i$ horizontally and bend the leg to rebalance the areas. This can be done so that the concave corner at $z'_i$ is resolved. We then say that $z'_i$ has been *convexified*. Fig. 5 shows an example of the process.

The vertex $z'_i$ has a concave corner in at most one of $F_{i-1}$ and $F_{i+1}$ In the first case we move $z'_i$ to the right in the second case we move $z'_i$ to the left. By symmetry, we only detail the second case, i.e, $z'_i$ has a concave corner in $F_{i+1}$.

Shifting $z'_i$ horizontally keeps the area of $F_i$ invariant, only the areas of $F_{i-1}$ and $F_{i+1}$ are affected by the shift. By shifting $z'_i$ a distance of $\delta$ to the left while

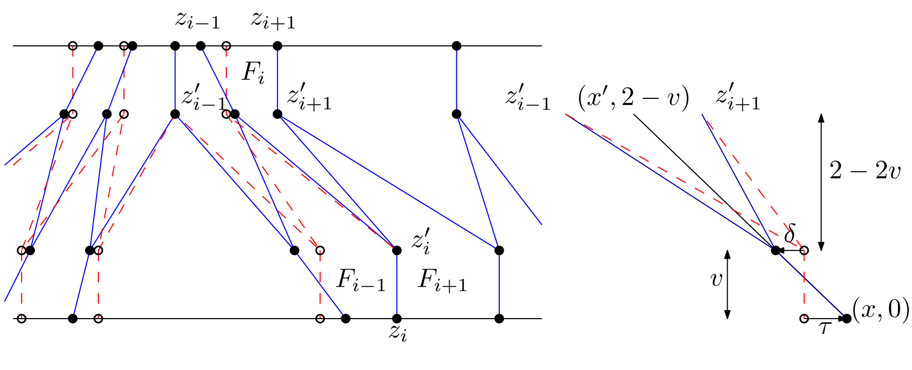


**Fig. 5.** Before and after convexifying $z'_i$. The dashed lines represent the original $Z'$.

keeping $z_i$ at its place the increase in area of $F_{i+1}$ is $\delta(2-v)/2$. To balance the increase we move $z_i$, the other end of the leg, to the right by an amount $\tau$, where $\tau v/2 = \delta(2-v)/2$. To make sure that the corners at $z'_i$ after shifting are convex we choose $\delta$ and $\tau$ so that the line connecting the new positions of $z_i$ and $z'_i$ contains the midpoint of $z'_{i-1}$ and $z'_{i+1}$. If the new position of $z_i$ is $(x, 0)$ and $(x', 2-v)$ is the midpoint of $z'_{i-1}$ and $z'_{i+1}$ then $v/(\tau+\delta) = 2/(x-x')$.

We do not want the shift of $z_i$ to introduce a crossing. We ensure this with a bound on $v$. For all $j$, let $T_j = Area(F_j)$ and this is the distance between $z_{j-1}$ and $z_{j+1}$ before shifting (since the height of the strip is 2). If $\tau \leq T_{i+1}$, then the leg $z'_i, z_i$ does not intersect leg $z'_{i+2}, z_{i+2}$. The absolute value of the slope of the leg $z'_i, z_i$ after convexifying $z'_i$ is less than $v/\tau$. The slope of the leg is also between the slopes of $z'_i, z'_{i-1}$ and $z'_i, z'_{i+1}$. The absolute value of these slopes is larger than $(2-2v)/(S/2)$ which is the minimum possible slope of a segment of $Z'$ in $R$. Define $T = \min_j T_j$. Hence, if $v/T < (2-2v)/(S/2) = 4(1-v)/S$, then $\tau < T$. We thus have an inequality that we want to be true for $v$:

$$v \leq \frac{4T}{S+4T}. \tag{B1}$$

Observe that convexifying $z'_{i+1}$ may require a shift of $z'_{i+1}$ by $\delta'$ (and a compensating shift of $z_{i+1}$ by $\tau'$) after $z'_i$ has been convexified. However, if $v \leq 1/4$, then balancing area and (B1) imply $\frac{1}{4}T > v\tau' = \delta'(2-v) \geq \delta'\frac{7}{4}$ whence $\delta' \leq T/7$. This shows that $z'_{i+1}$ stays on the right side of the old midpoint of $z'_{i-1}$ and $z'_{i+1}$ so that the corners at $z'_i$ stay convex.

The next step of the construction is to place equidistant points on each of the legs. The segments between two consecutive points on the leg $z'_i, z_i$ will serve as sides for quadrangles of the quadrangular subdivision of $F_{i-1}$ and $F_{i+1}$. Specifically, a leg $z'_i, z_i$ with $i$ odd is subdivided into $k-1$ segments of equal length and a leg $z'_i, z_i$ with $i$ even is subdivided into $m-k$ segments. Recall that $k$ is the number of rows in $A^t$. For the partition of $F_i$ into 4-gons with the prescribed areas we proceed inductively as in Lemma 2. We again need a partition lemma, whose proof is omitted due to space constraints.

**Lemma 4.** *Consider a convex 5-gon $F$ as shown in Fig. 6(a). Let $\alpha$, $\beta$, $\gamma$ be positive numbers with $\alpha+\beta+\gamma = Area(F)$. If $\alpha > Area(\square p_0, q_0, q_j, p_j)$, $\beta > Area(\triangle p_j, r, p_{j+1})$, $\gamma > Area(\triangle q_j, q_{j+1}, r)$, then there exists $p \in F$ such that $\alpha = Area(\pentagon p_0, q_0, q_j, p, p_j)$, $\beta = Area(\square p_j, p, r, p_{j+1})$, $\gamma = Area(\square q_j, q_{j+1}, r, p)$.*

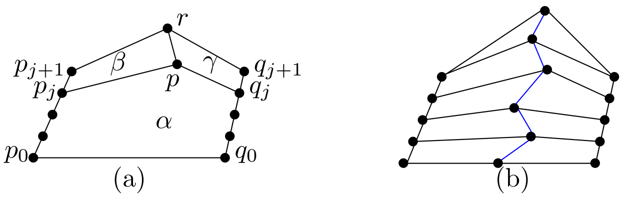


**Fig. 6.** (a) The $\alpha$, $\beta$ and $\gamma$ partition of $F$. (b) A final partition of $F_i$.

To ensure that the conditions for Lemma 4 are satisfied throughout the inductive partition of the regions $F_i$ we need to bound $v$. Let $M = \min_{i,j} A_{i,j}$ be the minimum value in the table. Recall that $S/2$ is the width of $R$, and the $y$-distance of $p_j$ and $p_{j+1}$ is at most $v$. Hence, $vS/2$ is a generous upper bound on $Area(\triangle p_j, r, p_{j+1})$, $Area(\triangle q_j, q_{j+1}, r)$, and $Area(\square p_0, q_0, q_j, p_j)$. We ensure that these areas are less than $\beta$, $\gamma$, and $\alpha$ respectively by requiring

$$v < \frac{2M}{S} \tag{B2}$$

**Theorem 2.** *Let $A$ be an $m \times n$ table of non-negative numbers $A_{i,j}$. Let $R$ be a rectangle of width $w$ and height $h$ such that $w \cdot h = \sum_{i,j} A_{i,j}$. Then there exists a non-degenerate cartogram of $A$ in $R$ such that every face in the cartogram is convex. The construction requires $O(mn)$ arithmetic operations.*

*Proof.* The steps of the construction are:

- Construct the table cartogram with degeneracies.
- Compute the bounds and fix an appropriate value for $v$, compute the skeleton $G'$ and its regions $F_i$, and convexify the legs in order of increasing index.
- Subdivide each of the regions $F_i$ into convex 4-gons (and two triangles).
- Remove the edges of the zig-zag to get the cells of the middle row as unions of two triangles, which must generate convex quadrangles, since these triangles are contained in rectangle $R$ and the common side of a pair of triangles connects opposite sides of $R$.

All can be done with $O(mn)$ arithmetic operations. Regarding the degeneracies, however, there is an issue that remains. To break $A$ into $A^t$ and $A^b$, we split row $k$ so that the last row of $A^t$ is a $\lambda$-fraction of row $k$ from $A$ while the rest of this row becomes the first row of $A^b$. Degeneracies occur if $\lambda = 1$. However, rather than splitting row $k$ in this case, we can treat cells of row $k$ as generic cells and assign a section of a leg to each of them. The construction is almost as before. Two details have to be changed. The first partition of each $F_i$ into three pieces now produces two 4-gons and a 5-gon, before (see Fig. 6(b)) we had two triangles and a 5-gon in this step. The other change is that we don't remove zig-zag edges belonging to $Z'$ to merge triangles to 4-gons at the end of the construction. □

Instead of just knowing that there are no degeneracies, it would be nice to have a lower bound on the *feature size*, that is the minimum side-length of a 4-gon in the table cartogram. The segments subdividing the legs have length at least $v/m$. Because these leg segments have length at most $v$ and $vS/2 < M$ (by (B2)), the opposite edges in a generic 4-gon (the blue edges in Fig. 6(b)) have length at least $v$. However, the triangles whose composition creates the 4-gons representing cells of row $k$ can have area smaller than $M$. These triangles may have area $\hat{\lambda}M$ where $\hat{\lambda} = \min\{\lambda, 1-\lambda\}$. This may lead to a very small feature size. To improve on this, another degree of freedom in the construction can be used. Instead of breaking each cell $A_{k,j}$ into a $\lambda$ and a $1-\lambda$ fraction, we can use individual values $\lambda_j$ to define $A^t_{k,j} = \lambda_j A_{k,j}$. The choice of the values $\lambda_j$ must satisfy two conditions: (1) $\sum_j \lambda_j A_{k,j} = \lambda S_k = \lambda \sum_j A_{k,j}$ and (2) if $\lambda_i = 0$ and $\lambda_j = 1$ then $|i - j| > 1$. By choosing most of the $\lambda_j$ to be 0 or 1, and avoiding degeneracies, we may be able to have a substantial improvement in feature size.

## 3 Generalizations

We generalize the notion of "area" by specifying the weight of a region as an integral over some density function $w : R \to \mathbb{R}^+$. The density function should be *positive*, meaning that the integrals over triangular regions with nonempty interiors exist and are positive. The following generalizes Lemma 1 for any positive density function, allowing us to compute cartograms on weighted $\mathbb{R}^2$.

**Lemma 5 (Weighted Triangle Lemma).** *Let $\triangle abc$ be a triangle and $w : \triangle abc \to \mathbb{R}^+$ be a positive density function on $\triangle abc$. Let $Area(\triangle abc)$ be the $w$-weighted area of the triangle $\triangle abc$. Given three non-negative real numbers $\alpha$, $\beta$, $\gamma$, where $\alpha + \beta + \gamma = Area(\triangle abc)$, there exists a unique point $p$ inside $\triangle abc$ such that $Area(\triangle pbc) = \alpha$, $Area(\triangle apc) = \beta$, and $Area(\triangle abp) = \gamma$.*

We now discuss some scenarios where the outerface of the cartogram has a more general shape. The following theorem considers the case when the outerface is a convex quadrangle $\square pqrs$. In such a case, we use a binary search to find the zigzag path that starts at $q$ and ends at $r$ or $s$.

**Theorem 3.** *Let $A$ be an $m \times n$ table of non-negative numbers. Let $\square pqrs$ be an arbitrary convex quadrilateral with area equal to the sum, $S$, of the numbers of $A$. Then there exists a cartogram of $A$ in $\square pqrs$ (with degeneracies).*

Next, we show how to compute a table cartogram inside a circle. A *circular triangle* $\bigcirc\!\!\!\!\triangle abc$ is a region in the plane bounded by three circular arcs (called *arms*) that pairwise meet at the points $a$, $b$, and $c$ (called *vertices*), such that for every vertex $v \in \{a, b, c\}$ and for every point $x$ on the arc that is not incident to $v$, one can draw a circular arc between $v$ and $x$ inside $\bigcirc\!\!\!\!\triangle abc$ that does not cross the boundary of $\bigcirc\!\!\!\!\triangle abc$. An arm is *convex* if the straight line joining any two points on the arm is interior to the region bounded by the triangle. Otherwise, the arm is *concave*. We distinguish four types of circular triangles, see Figs. 7(a–d). We generalize Lemma 1 for circular triangles, i.e., given a circular triangle, one can split it into three other circular triangles with prescribed areas, and find the following generalization of Theorem 1.

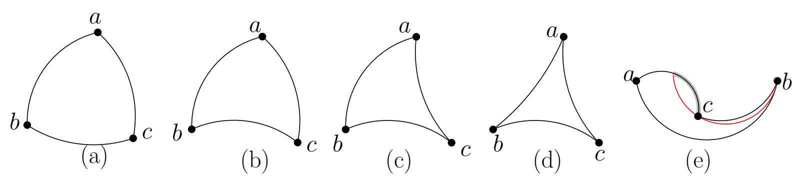


**Fig. 7.** (a–d) A circular triangle of Type $i$, $0 \leq i \leq 3$, i.e., a circular triangle with $i$ concave arms. (e) A region bounded by circular arcs, but not a circular triangle.

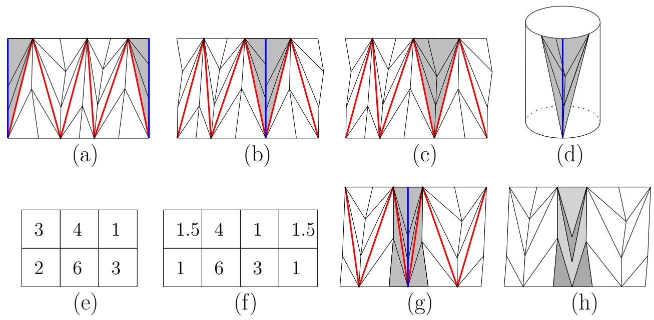


**Fig. 8.** Cylindrical cartogram construction. (a–d) $n$ is even, (e–h) $n$ is odd. Note that when $n$ is even, the faces are convex quadrilaterals. However, when $n$ is odd, the faces with areas from the leftmost column of $A$ may be concave hexagons.

**Theorem 4.** *Let $A$ be an $m \times n$ table of non-negative numbers. Let $C$ be an arbitrary circle with area equal to the sum of the numbers of $A$. Then there exists a cartogram of $A$ in $C$ where every face is a circular triangle.*

We now show that one can compute a cartogram of a table on the surface of a sphere. There are several well studied area preserving map projection techniques. Here we use Lambert's cylindrical equal-area projection. The construction used is shown in Fig. 8.

**Theorem 5.** *Every $m \times n$ table of non-negative numbers admits a cartogram on a sphere.*

## 4 Conclusions and Future Work

We have presented a simple constructive algorithm that realizes any table inside a rectangle in which each cell is represented by a convex quadrilateral with its prescribed weight. If all weights are strictly positive, then we can also obtain non-degenerate realizations. This method can be further extended to realize any table inside an arbitrary convex quadrilateral, inside a circle using circular arcs, or even on a sphere. From a practical point of view, the cartograms obtained by our method may not be visually pleasing, but by using additional straightforward heuristics that improve the visual quality while keeping the areas the same, we can obtain cartograms of practical relevance, as shown in Figs. 1 and 3. Our theoretical solution plays a vital role in this context, since heuristics used directly may get stuck, being unable to obtain the correct areas. Whether there exists a method that can gradually change the areas to provably obtain the correct areas remains an interesting open problem. It would also be interesting to examine table cartograms for other types of tables, such as triangular or hexagonal grids. From a theoretical point of view, finding algorithms for table cartograms on a sphere with less distortion, and generalizing our result to 3D table cartograms (inside a box) are further interesting open problems.

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