Approximate Proximity Drawings☆

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Abstract

We introduce and study a generalization of the well-known region of influence proximity drawings, called *$(\varepsilon_1, \varepsilon_2)$-proximity drawings*. Intuitively, given a definition of proximity and two real numbers $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$, an $(\varepsilon_1, \varepsilon_2)$-proximity drawing of a graph is a planar straight-line drawing Γ such that: (i) for every pair of adjacent vertices u, v, their proximity region "shrunk" by the multiplicative factor $\frac{1}{1+\varepsilon_1}$ does not contain any vertices of Γ; (ii) for every pair of non-adjacent

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vertices u, v, their proximity region "expanded" by the factor $(1 + \varepsilon_2)$ contains some vertices of Γ other than u and v. In particular, the locations of the vertices in such a drawing do not always completely determine which edges must be present/absent, giving us some freedom of choice. We show that this generalization significantly enlarges the family of representable planar graphs for relevant definitions of proximity drawings, including Gabriel drawings, Delaunay drawings, and β-drawings, even for arbitrarily small values of ε_1 and ε_2. We also study the extremal case of $(0, \varepsilon_2)$-proximity drawings, which generalize the well-known weak proximity drawing paradigm.

Keywords: Graph drawing, Gabriel graphs, Beta skeletons, Delaunay triangulations, Approximation

1. Introduction and overview

Proximity drawings are straight-line drawings of graphs where any two adjacent vertices are deemed to be close according to some proximity measure, while any two non-adjacent vertices are far from one another by the same measure. Different definitions of proximity give rise to different types of proximity drawings. In the *region of influence* based proximity drawings two vertices u and v are adjacent if and only if some regions of the plane, defined by using the coordinates of u and v, are *empty*, i.e. they do not contain any vertices of the drawing other than, possibly, u and v.

For example, the *Gabriel disk* of two points u and v in the plane is the closed disk having u and v as its antipodal points (cf. Fig. 1(a)) and a *Gabriel drawing* is a planar straight-line drawing Γ such that any two vertices in Γ are connected by an edge if and only if their Gabriel disk is empty of other vertices. Note that any such drawing Γ, viewed as a geometric graph in the plane, coincides with the so-called *Gabriel graph* of the points forming the vertex set of Γ.

A generalization of the Gabriel disk is the so-called *β-region of influence* (cf. Fig. 1(b) and (c)): For a given value of β such that $1 \leq \beta \leq \infty$, the β-region of influence of two vertices u and v having Euclidean distance $d(u, v)$ is the intersection of the two disks of radius $\frac{\beta d(u,v)}{2}$, centered on the line through u and v, one containing u and touching v, the other containing v and touching u (hence the β-region for $\beta = 1$ is the Gabriel disk). Given a value of β, a straight-line drawing Γ is a *β-drawing* if and only if for any edge (u, v) in Γ the β-region of influence of u and v is empty of other vertices, that is, Γ coincides with the so-called *β-skeleton* of the points forming the vertices of Γ.

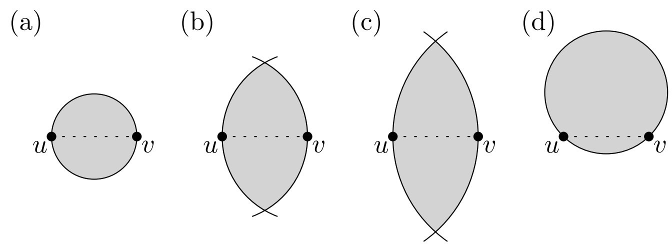

Figure 1: Two points u and v. Indicated by shading: (a) their Gabriel disk (which coincides with their β-region for $\beta = 1$), (b) their β-region for $\beta = 2$, (c) their β-region for $\beta = 3$, and (d) one of their Delaunay disks.

Delaunay drawings use a definition of proximity that extends the one used for Gabriel drawings. Namely, the *Delaunay disks* of two vertices u and v are the closed disks having $\overline{uv}$ as a chord (the Gabriel disk is therefore a particular Delaunay disk, cf. Fig. 1(d)). In a *Delaunay drawing* Γ an edge (u, v) exists if and only if at least one of the Delaunay disks of u and v is empty of other vertices, that is, Γ coincides with the so-called *Delaunay graph* of its vertex set. Note that the Delaunay graph of a point set P is not a triangulation of P if, for example, more than three points in P lie on a common circle that does not contain other points of P in its interior.

As is not hard to imagine, by changing the definition of region of influence, the combinatorial properties of those graphs that admit a certain type of proximity drawing can change significantly. For example, it is known that not all trees having vertices of degree four admit a Gabriel drawing [5] while they have a β-drawing for $1 < \beta \leq 2$ [15]. It should be noted, however, that despite the many papers published on the topic, full combinatorial characterization of proximity drawable graphs remains an elusive goal for most types of regions of influence. The interested reader is referred to [7, 14, 17] for more references and results on these topics. As a general tendency, using the region of influence based proximity rules recalled above only very restricted families of graphs can be represented. In this paper, we propose a generalization of these rules and show that it considerably extends the families of representable graphs.

1.1. Problem and results

In order to overcome the restrictions on the families of graphs representable as region of influence based proximity drawings, we study graph visualizations that are "good approximations" of these proximity drawings. We want drawings where adjacent vertices are relatively close to each other while non-adjacent vertices are relatively far apart. The idea is to use slightly smaller regions of influence to justify the existence of an edge and slightly larger regions of influence to justify non-adjacent vertices. Note that, since different regions are used to justify the existence and non-existence, respectively, of an edge, it may happen that for a particular pair of vertices in a drawing we actually have the choice to either draw an edge between these vertices or not. In contrast, once the vertex set of a drawing is fixed, the usual proximity rules completely determine which edges must be present/absent in the drawing. This key difference is one of the reasons why larger families of graphs can be represented in the framework presented here. In the following we focus on the representability of various types of planar graphs as this helps us to emphasize the connection with the original proximity rules that necessarily yield plane drawings. Note, however, that our new framework can also represent non-planar graphs.

Now, to describe the modified regions of influence more formally, let D be a disk with center c and radius r, and let ε_1 and ε_2 be two non-negative real numbers. The ε_1-*shrunk disk of* D is the disk centered at c and having radius $\frac{r}{1+\varepsilon_1}$; the ε_2-*expanded disk of* D is the disk centered at c and having radius $(1+\varepsilon_2)r$. An $(\varepsilon_1, \varepsilon_2)$-*proximity drawing* is a planar straight-line proximity drawing where the region of influence of two adjacent vertices is defined by using ε_1-shrunk disks, while the region of influence of two non-adjacent vertices uses ε_2-expanded disks. Sometimes we will simply refer to such a drawing as an *approximate* proximity drawing.

To illustrate the above definitions, note that all planar graphs (actually, all graphs) with at least one edge or at least three vertices have an $(\varepsilon_1, \varepsilon_2)$-proximity drawing for sufficiently large values of $\varepsilon_1, \varepsilon_2$. For example, every planar straight-line drawing Γ of such a graph is an (∞, ∞)-Gabriel drawing since an ∞-shrunk Gabriel disk reduces to a point (and thus the ∞-shrunk disk of every edge in Γ is empty) and an ∞-expanded Gabriel disk is the whole plane (and thus the ∞-expanded disk of any pair of non-adjacent vertices of Γ contains a third vertex, if the graph contains at least three vertices). At the other extreme, a $(0,0)$-Gabriel drawing is a Gabriel drawing, since a 0-shrunk Gabriel disk is a Gabriel disk and so is a 0-expanded Gabriel disk. Hence, not all planar graphs admit a $(0,0)$-Gabriel drawing [5].

Based on this observation, our main target is to establish values of ε_1 and of ε_2 that make it possible to compute $(\varepsilon_1, \varepsilon_2)$-proximity drawings for meaningful families of planar graphs and embedded planar graphs. To this end, recall that, for a planar graph G, a *(planar) embedding* of G specifies which face of G is the outer face and, for every vertex v of G, the circular order of the edges adjacent to

v. An *embedded planar graph* is a planar graph together with an embedding. If a planar straight-line drawing Γ of G has the same embedding, it is said to *maintain* (or *preserve*) the embedding of G; in this case we shall say that Γ is an *embedding preserving drawing* of G.

This paper is structured as follows. After summarizing some related work, the next three sections are devoted to approximate Gabriel drawings (Section 2), approximate β-drawings (Section 3) and approximate Delaunay drawings (Section 4). We show that each of these types of drawings allows, for any embedded planar graph G, an embedding preserving $(\varepsilon_1, \varepsilon_2)$-proximity drawing of G, as long as the parameters ε_1 and ε_2 are both strictly positive. Moreover, we establish that these results are, in a sense, tight by exhibiting embedded planar graphs that do not have an embedding preserving $(\varepsilon_1, \varepsilon_2)$-proximity drawing with either $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$. Then, in Section 5, we study $(0, \varepsilon_2)$-proximity drawings which, as explained below, make it possible to express different proximity conventions in a unified framework. In particular, we study $(0, \varepsilon_2)$-Gabriel drawings of outerplanar graphs, extending previous results of Di Battista et al. [8] and Lenhart and Liotta [15]. We conclude in Section 6 where we also mention some open problems.

We emphasize that the main contribution of this paper is in introducing the concept of $(\varepsilon_1, \varepsilon_2)$-proximity drawing and in proving the existence of $(\varepsilon_1, \varepsilon_2)$-proximity drawings for relevant families of graphs. Hence, we shall not analyze in detail the time complexities of our algorithms; it is not hard to see, however, that our drawing techniques all require polynomial time when adopting the real RAM model of computation.

1.2. Related work

Several generalizations, variants, and relaxations of proximity drawings have been defined in the literature such as, for example, k-localized Delaunay triangulations, approximate minimum spanning trees, and witness proximity drawings. While the interested reader can, for example, use [2, 4, 9, 16] to find more references on these topics, in this introduction we will particularly recall the notion of weak proximity drawings [8], that are more closely related with $(\varepsilon_1, \varepsilon_2)$-proximity drawings.

In a weak proximity drawing, the region of influence of any pair of adjacent vertices must be empty, while no condition is given for the non-adjacent pairs. Hence, weak proximity drawings guarantee visual closeness of groups of edge-related vertices but do not ensure that unrelated vertices are far apart. In contrast, $(\varepsilon_1, \varepsilon_2)$-proximity drawings guarantee some relative closeness of the adjacent pairs of vertices *and* some relative separation of the non-adjacent pairs for small values of ε_1 and ε_2.

Note that a weak proximity drawing is a $(0, \infty)$-proximity drawing and that a proximity drawing in the traditional sense is a $(0, 0)$-proximity drawing. Therefore, $(0, \varepsilon_2)$-proximity drawings make it possible to study proximity drawability in a unified framework: as the value of ε_2 increases, $(0, \varepsilon_2)$-proximity drawings approach weak proximity drawings. Several questions can be asked within this unifying framework. For example, not all trees have a Gabriel drawing, while all trees have a weak Gabriel drawing. What is the minimum threshold value such that if ε_2 is larger than this threshold all trees are drawable? Theorem 4 answers this question.

Di Battista et al. [8] show that every tree admits a weak proximity drawing using β-regions of influence for β less than 2, while it is NP-hard to determine whether a tree with vertex degree four has a weak β-drawing for $\beta = \infty$. In contrast, Theorem 2 proves that for any finite positive values of $\varepsilon_1, \varepsilon_2$ all planar graphs admit an $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing for all values of β such that $1 \leq \beta \leq \infty$.

2. Approximate Gabriel drawings

Let Γ be a planar straight-line drawing of a graph and let $\varepsilon_1, \varepsilon_2$ be two non-negative numbers. Let u, v be any two vertices of Γ and let $D(u, v)$ be the Gabriel disk of u, v (that is, the smallest disk containing u and v, cf. Fig. 1(a)). We say that Γ is an $(\varepsilon_1, \varepsilon_2)$-*Gabriel drawing* if: (i) for every edge (u, v) of Γ the ε_1-shrunk disk of $D(u, v)$ is empty (i.e. it does not contain any vertex of Γ other than, possibly, u and v); and (ii) for every pair of non-adjacent vertices u, v of Γ, the ε_2-expanded disk of $D(u, v)$ is not empty (i.e. it contains some vertex w of Γ other than u and v). Note that a Gabriel graph is a special case of an $(\varepsilon_1, \varepsilon_2)$-Gabriel drawings, namely the one in which $\varepsilon_1 = \varepsilon_2 = 0$.

Fig. 2 is an example of an $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing for $\varepsilon_1 = 0$ and $\varepsilon_2 = 0.7$. The drawing is not a Gabriel drawing; for example, the dotted disk in the figure is a Gabriel disk, while the solid one is its 0.7-expanded version. Note that a Gabriel drawing does not exist for the tree of Fig. 2 [5].

In order to establish values of $\varepsilon_1, \varepsilon_2$ that allow an $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing of every planar graph, we start by considering the extremal cases that either $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, or $\varepsilon_1 > 0$ and $\varepsilon_2 = 0$. The next two lemmas study the relationship between embedding preserving $(\varepsilon_1, \varepsilon_2)$-Gabriel drawings of this type and Gabriel graphs. We say that an embedded planar graph G is *Gabriel drawable* if there exists a Gabriel graph Γ such that Γ is an embedding preserving drawing of G.

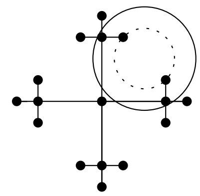

Figure 2: A (0,0.7)-Gabriel drawing of a tree that does not have a $(0,0)$-Gabriel drawing.

Lemma 1. *Let G be an embedded maximal planar triangulation and let ε_2 be any given real number such that $\varepsilon_2 \geq 0$. G has an embedding preserving $(0, \varepsilon_2)$-Gabriel drawing if and only if G is Gabriel drawable.*

Proof. First note that if G has an embedding preserving Gabriel drawing Γ, then Γ is also a $(0, \varepsilon_2)$-Gabriel drawing for any $\varepsilon_2 > 0$. So assume, conversely, that G has an embedding preserving $(0, \varepsilon_2)$-Gabriel drawing Γ for some $\varepsilon_2 > 0$. Let V denote the vertex set of Γ. Note that the Gabriel graph $GG(V)$ induced by the points in V contains Γ as a subgraph. Since the Gabriel graph of a point set is a planar geometric graph [14] and G is a maximal planar triangulation, it follows that $GG(V)$ coincides with Γ. $\square$

Recall that a *separating three-cycle* in an embedded planar triangulation is a cycle of length 3 such that at least one vertex lies inside the region enclosed by the cycle and at least one vertex lies outside this region. In addition, note that every embedded planar triangulation with a separating three-cycle does not have an embedding preserving Gabriel drawing. This follows from the fact that any vertex inside the triangle representing the three-cycle in an embedding preserving drawing is contained in the Gabriel disk for at least one pair of vertices of this triangle. As a consequence, Lemma 1 implies the following.

Corollary 1. *There exist embedded planar graphs that do not have an embedding preserving $(0, \varepsilon_2)$-Gabriel drawing, for any $\varepsilon_2 \geq 0$.*

The proof of the next lemma focuses on pairs of non-adjacent vertices and follows a similar argument to the one of Lemma 1.

Lemma 2. *Let T be an embedded tree and let ε_1 be any given real number such that $\varepsilon_1 \geq 0$. T has an embedding preserving $(\varepsilon_1, 0)$-Gabriel drawing if and only if T is Gabriel drawable.*

Proof. First note that if T has an embedding preserving Gabriel drawing Γ, then Γ is also a $(\varepsilon_1, 0)$-Gabriel drawing for any $\varepsilon_1 > 0$. Next assume, conversely, that T has an embedding preserving $(\varepsilon_1, 0)$-Gabriel drawing Γ for some $\varepsilon_1 > 0$. Let V be the vertex set of Γ and let $GG(V)$ be the Gabriel graph induced by the points in V. Note that $GG(V)$ is a subgraph of Γ. Since the Gabriel graph of a point set is a connected graph [14] and Γ is a tree, it follows that $GG(V)$ coincides with Γ. $\square$

Lemma 2 and the characterization of which trees admit a Gabriel drawing [5] immediately imply the following.

Corollary 2. *There exist embedded planar graphs that do not have an embedding preserving $(\varepsilon_1, 0)$-Gabriel drawing, for any $\varepsilon_1 \geq 0$.*

Motivated by Corollaries 1 and 2, we move our attention to $(\varepsilon_1, \varepsilon_2)$-Gabriel drawings where both $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. We prove that one can compute a drawing that approximates a Gabriel drawing for every planar graph, provided that the Gabriel region is scaled down for the edges and is scaled up for the non-adjacent pairs of vertices by any arbitrarily small chosen amount.

Lemma 3. *Let $\varepsilon_1, \varepsilon_2$ be two real numbers such that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Every embedded planar graph with at least one edge has an embedding preserving $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing.*

Proof. Let G be a planar graph with a given planar embedding. If necessary, we add edges to G that respect the planar embedding to turn it into an embedded planar triangulation G'. Note that it is always possible to add edges so that in G' at least one of the edges of the outer triangle is also contained in G. Also note that our construction of an approximate Gabriel drawing of G below will be such that the ε_2-expanded Gabriel disk associated to any of those added edges is not empty.

Now, let $v_1, \dots, v_n$ be a canonical ordering (as defined by de Fraysseix, Pach, and Pollack [6]) of the vertices of the embedded triangulated graph G' so that (v_1, v_2) is an edge of G. Let G_i be the subgraph of G induced by the vertices in $V_i = \{v_1, v_2, \dots, v_i\}$. We show how to construct a drawing Γ_i of G_i by induction so that, for all $i \geq 2$, (a) Γ_i is an embedding preserving $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing of G_i; and (b) all vertices in V_i that lie on the outer face of Γ_i are *horizontally visible* from the right, that is, the horizontal ray emanating from such a vertex to the right does not intersect any edge of Γ_i. Clearly we can satisfy these properties for $i = 2$, since (v_1, v_2) is an edge of G, by drawing v_1 and v_2 at points $(0, 1)$ and $(1, n)$, respectively.

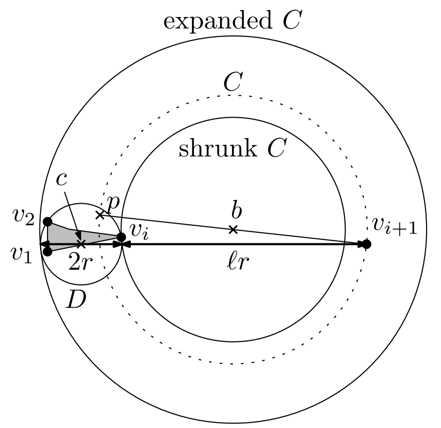

Figure 3: Proof of Lemma 3.

Next, assuming we have Γ_i for some $i \geq 2$, we show how to construct Γ_{i+1}. Let N_i be the vertices in V_i that are adjacent to v_{i+1}. These vertices are consecutive on the outer face of Γ_i since $v_1, v_2, \ldots, v_n$ is a canonical ordering. Let the y-coordinate of v_{i+1} be $y(v_{i+1}) \in (\min_{v \in N_i} y(v), \max_{v \in N_i} y(v))$ (any coordinate in the range will suffice) where $y(v_1) = 1$ and $y(v_2) = n$. We place vertex v_{i+1} far enough to the right so that for every $v_j, v_k \in V_i$, (i) an edge from v_{i+1} to v_j is permitted by the ε_1-shrunk Gabriel disk $D(v_{i+1}, v_j)$ (i.e. the shrunken disk is empty); (ii) if (v_{i+1}, v_j) is *not* an edge in G then the ε_2-expanded Gabriel disk $D(v_{i+1}, v_j)$ prevents the edge (i.e. the expanded disk contains a vertex); and (iii) v_{i+1} does not lie in the ε_1-shrunk Gabriel disk $D(v_j, v_k)$.

Let D be the smallest disk centered on y-coordinate $y(v_{i+1})$ that encloses Γ_i. Let c be the center of D and r be the radius of D. Let ℓr be the (still to be determined) distance of v_{i+1} from the rightmost point of D. We choose ℓ so that for every $p \in D$, if C is the disk with diameter $\overline{pv_{i+1}}$ (in fact, C can be any disk with chord $\overline{pv_{i+1}}$), (I) the ε_1-shrunk C does not intersect D (implying Property (i)), and (II) the ε_2-expanded C contains D (implying Property (ii)). Let b be the center of C. Since b is on the perpendicular bisector of $\overline{pv_{i+1}}$, $d(b,p) \geq (\ell/2)r$. Refer to Fig. 3.

Property (I) is equivalent to $\frac{d(b,p)}{d(b,c)-r} < 1 + \varepsilon_1$. By the triangle inequality, $d(b,p) \leq d(b,c) + d(c,p) \leq d(b,c) + r$. Thus, $\frac{d(b,p)}{d(b,c)-r} \leq \frac{d(b,p)}{d(b,p)-2r} = 1 + \frac{2r}{d(b,p)-2r} \leq 1 + \frac{2r}{(\ell/2)r-2r} = 1 + \frac{4}{\ell-4}$. Property (II) is equivalent to $\frac{d(b,c)+r}{d(b,p)} < 1 + \varepsilon_2$. By the triangle inequality, $d(b,c) \leq d(b,p) + d(p,c) \leq d(b,p) + r$. Thus, $\frac{d(b,c)+r}{d(b,p)} \leq \frac{d(b,p)+2r}{d(b,p)}$

$= 1 + \frac{2r}{d(b,p)} \leq 1 + 4/\ell$. If we choose ℓ large enough so that $4/(\ell - 4) < \varepsilon_1$ and $4/\ell < \varepsilon_2$ then we satisfy both Properties (i) and (ii). By choosing $\ell > \sqrt{2} - 1$, we satisfy Property (iii) since the union of 0-shrunk (and hence all ε_1-shrunk) disks of line segments in a disk D^* is contained in the $\sqrt{2}$-expanded version of D^*. In addition, we choose ℓ large enough so that no edge from v_{i+1} to v_j, $j \leq i$, crosses any (already drawn) edge in Γ_i. Thus, we ensure that Γ_{i+1} respects the given embedding and is a $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing of G_{i+1}. □

The results in this section can be summarized as follows.

Theorem 1. *Let G be an embedded planar graph with at least one edge. For any given values of $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, G admits an embedding preserving $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing. Also, there exist embedded planar graphs that do not have an embedding preserving $(0, \varepsilon_2)$-Gabriel drawing and embedded planar graphs that do not have an embedding preserving $(\varepsilon_1, 0)$-Gabriel drawing.*

This result for Gabriel drawings raises the question of which other proximity regions allow all planar graphs to be drawn as $(\varepsilon_1, \varepsilon_2)$-proximity drawings for any arbitrarily small positive values of ε_1 and ε_2. The next two sections examine two such proximity regions.

3. Approximate β-drawings

Let $\varepsilon_1, \varepsilon_2$ be two non-negative numbers and let β be a real number such that $\beta \geq 1$. Let Γ be a planar straight-line drawing of a graph and let u, v be any two vertices of Γ. The β-region of influence of u and v (cf. Fig. 1(b) and (c)), denoted as $\beta(u, v)$, is the intersection of two disks D_u and D_v such that: (i) both D_u and D_v are centered on the line through u, v; (ii) both D_u and D_v have radius $\frac{\beta d(u,v)}{2}$, where $d(u, v)$ is the Euclidean distance between u and v; D_u contains v and D_v contains u; and (iii) the circumference of D_u contains u and the circumference of D_v contains v. The ε_1-*shrunk* β-*region of influence* of u and v is defined as the intersection of the ε_1-shrunk disk of D_u with the ε_1-shrunk disk of D_v. Similarly, the ε_2-*expanded* β-*region of influence* of u and v is the intersection of the ε_2-expanded disks of D_u and D_v.

We say that Γ is an $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing if: (i) for every edge (u, v) of Γ the ε_1-shrunk β-region of influence of u and v is empty; and (ii) for every pair of non-adjacent vertices u, v of Γ, the ε_2-expanded β-region of influence of u and v is not empty. Note that a $(0, 0)$-β-drawing Γ is just a β-drawing, i.e. Γ coincides with the β-skeleton its vertex set. Later we will use the fact that, for any $\beta_1 \geq \beta_2 \geq 1$

and any nonempty, finite point set P, the β_1-skeleton of P is a subgraph of the β_2-skeleton of P.

Note that, by definition, an $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing with $\beta = 1$ is an $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing. Therefore, it is probably not surprising that, using an argument completely analogous to the one employed in the proof of Lemma 1, one can show that, for any $\varepsilon_2 > 0$ and any $\beta > 1$, an embedded maximal planar triangulation G has an embedding preserving $(0, \varepsilon_2)$-β-drawing if and only if G has an embedding preserving β-drawing. Similarly, for any $\varepsilon_1 > 0$ and any $\beta > 1$, an embedded tree T has an embedding preserving $(\varepsilon_1, 0)$-β-drawing if and only if G has an embedding preserving β-drawing. The fact that, for all $\beta > 1$, there exist embedded maximal planar triangulations (in particular those containing a separating three-cycle) and also embedded trees [5] that do not have an embedding preserving β-drawing then implies that not all embedded planar graphs admit an $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing that respects the given embedding, when either ε_1 or ε_2 is set to 0. On the other hand, we can extend Lemma 3 to all values of $\beta > 1$. The proof technique is similar to the one in Lemma 3.

Lemma 4. *Let $\varepsilon_1, \varepsilon_2$ be two real numbers such that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and let β be a real number such that $\beta \geq 1$. Every embedded planar graph with at least one edge has a $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing that maintains the given embedding.*

Proof. The construction given in the proof of Lemma 3 works here with two modifications: To establish Property (i) for β-drawings, we consider the largest possible value ($\beta = \infty$). It suffices to choose ℓ large enough so that the ε_1-shrunk β-region of influence of any point p contained in D and v_{i+1} is empty. Thus, since D has diameter $2r$, we require $\frac{1}{2}(\ell r - \frac{\ell r}{1+\varepsilon_1}) > 2r$, which occurs if $\ell\varepsilon_1/(1+\varepsilon_1) > 4$. Note that Property (ii) for β-drawings follows without any modifications by choosing $4/\ell < \varepsilon_2$ in view of the fact that the Gabriel disk for a pair of points is always contained in the β-region for this pair of points for any $\beta > 1$. To establish Property (iii), v_{i+1} must be outside the union of the ε_1-shrunk β-regions for edges in Γ_i. It suffices to place v_{i+1} outside the union of 0-shrunk $(\beta = \infty)$-regions for edges in Γ_i. Since none of the edges in Γ_i are vertical, this union does not intersect the region $(x, \infty) \times (1, n)$ for some large enough x. It suffices to choose ℓ so that v_{i+1} has x-coordinate larger than x. □

We can summarize the discussion of this section as follows.

Theorem 2. *Let G be an embedded planar graph with at least one edge. For any given values of $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and for any value of β such that*

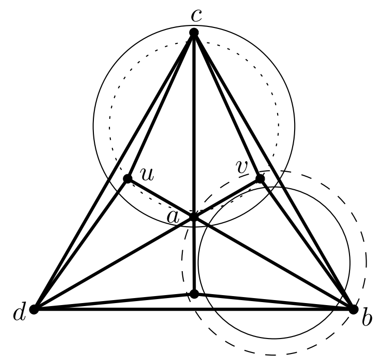

Figure 4: An $(\varepsilon_1, \varepsilon_2)$-Delaunay drawing for $\varepsilon_1 = 0.25$ and $\varepsilon_2 = 0.2$ of an embedded planar graph that does not have an embedding preserving Delaunay drawing.

$\beta \geq 1$, G admits an embedding preserving $(\varepsilon_1, \varepsilon_2)$-$\beta$-drawing. Also, there exist embedded planar graphs that do not have a $(0, \varepsilon_2)$-β-drawing and planar graphs that do not have a $(\varepsilon_1, 0)$-β-drawing that maintain the given embedding.

4. Approximate Delaunay drawings

Let Γ be a planar straight-line drawing of a graph and let $\varepsilon_1, \varepsilon_2$ be two non-negative numbers. Let u, v be any two vertices of Γ and let $\mathcal{D}(u, v)$ be the set of all disks in the plane that have $\overline{uv}$ as a chord. Let $\mathcal{D}_{\varepsilon_1}(u, v)$ be the set of the ε_1-shrunk disks of $\mathcal{D}(u, v)$ and let $\mathcal{D}_{\varepsilon_2}(u, v)$ be the set of the ε_2-expanded disks of $\mathcal{D}(u, v)$. Drawing Γ is an $(\varepsilon_1, \varepsilon_2)$-*Delaunay drawing* if: (i) for any two adjacent vertices u, v of Γ, there exists at least one empty disk in $\mathcal{D}_{\varepsilon_1}(u, v)$; and (ii) for any two non-adjacent vertices u, v of Γ, all disks of $\mathcal{D}_{\varepsilon_2}(u, v)$ contain some vertex of Γ other than u and v. Note that a Delaunay drawing is a special case of $(\varepsilon_1, \varepsilon_2)$-Delaunay drawings, namely the one in which $\varepsilon_1 = \varepsilon_2 = 0$.

Fig. 4 is an example of an $(\varepsilon_1, \varepsilon_2)$-Delaunay drawing for $\varepsilon_1 = 0.25$ and $\varepsilon_2 = 0.2$. In the figure, a Delaunay disk for the two vertices u and v is depicted (dotted). As can be seen, the ε_2-expanded version of this disk contains vertices a and c. In fact, the ε_2-expanded version of *any* Delaunay disk for u and v contains at least one of the vertices a or c and, therefore, u and v are not adjacent in the drawing. In contrast, there is no Delaunay disk for the two vertices a and b. In particular, the disk drawn dashed is not a Delaunay disk for this pair of points. The ε_1-shrunk version of this disk, however, is empty and, thus, justifies the existence of the edge with endpoints a and b in the drawing.

In the context of Delaunay drawings, a vertex set V is *degenerate* if either four or more co-circular vertices define a circle that does not contain another vertex in its interior, or there are three or more collinear vertices on the boundary of the convex hull (see, e.g. [12]). Note that if the vertex set of a Delaunay drawing Γ is not degenerate then Γ is necessarily a triangulation of the point set V forming its vertices, usually referred to as the *Delaunay triangulation* of V. Also note that, conversely, if a Delaunay drawing is a maximal planar triangulation then its vertex set cannot be degenerate.

Since Delaunay triangulations are among the most studied graphs in computational geometry, we start by investigating the relationship between $(\varepsilon_1, \varepsilon_2)$-Delaunay drawings and Delaunay drawings. In the proof of the next result we will use the following result of Dillencourt [11]: Let Γ be a Delaunay drawing with possibly degenerate vertex set V. Add, if necessary, edges to Γ to obtain a triangulation of V. Let Γ' denote the resulting drawing. Then, for any subset $U \subseteq V$, removing all vertices in U along with all edges incident to a vertex in U yields a drawing with at most $|U| - 2$ connected components that do not contain a vertex of the convex hull of V. Note that this immediately implies that the graph with the planar embedding of Fig. 4 does not admit an embedding preserving Delaunay drawing [11] since removing vertices a, b, c and d yields three connected components.

Lemma 5. *There exist embedded maximal planar graphs that do not admit an embedding preserving* $(\varepsilon_1, 0)$*-Delaunay drawing, for any value* $\varepsilon_1 \geq 0$.

Proof. Let G be the embedded maximal planar graph depicted in Fig. 4. As noted above, G does not have a Delaunay drawing and, hence, it does not have an embedding preserving $(0, 0)$-Delaunay drawing.

Now, suppose G has an $(\varepsilon_1, 0)$-Delaunay drawing Γ for some $\varepsilon_1 > 0$. Let V be the vertex set of Γ and let $DG(V)$ be the Delaunay graph of V. The graph $DG(V)$ is a subgraph of Γ since every non-edge of Γ is a non-edge of $DG(V)$. Clearly we can add exactly those edges of Γ that are not contained in $DG(V)$ to obtain a triangulation of V, namely Γ. But this contradicts Dillencourt's result above since, as noted above already, removing vertices a, b, c and d from Γ yields three connected components. $\square$

While Lemma 5 considers $(\varepsilon_1, 0)$-Delaunay drawings of maximal planar triangulations, one can wonder what happens with the other extreme, that is, with $(0, \varepsilon_2)$-Delaunay drawings. We use arguments similar to those in the proof of Lemma 1 to prove the following lemma and corollary.

Lemma 6. *Let G be an embedded maximal planar triangulation and let ε_2 be any given real number such that $\varepsilon_2 \geq 0$. G has an embedding preserving $(0, \varepsilon_2)$-Delaunay drawing if and only if G has a Delaunay drawing.*

Proof. First note that if G has an embedding preserving Delaunay drawing Γ, then Γ is also a $(0, \varepsilon_2)$-Delaunay drawing for any $\varepsilon_2 > 0$. Now suppose that, conversely, G has an embedding preserving $(0, \varepsilon_2)$-Delaunay drawing Γ for some $\varepsilon_2 > 0$. Let V be the vertex set of Γ and let $DG(V)$ be the Delaunay graph having V as its vertex set. Note that Γ is a subgraph of $DG(V)$. Since $DG(V)$ is a planar geometric graph and G is a maximal planar triangulation, it follows that $DG(V)$ coincides with Γ. Hence, $DG(V)$ is a Delaunay drawing of G. □

Corollary 3. *There exist embedded planar graphs that do not have an embedding preserving $(0, \varepsilon_2)$-Delaunay drawing, for any $\varepsilon_2 \geq 0$.*

By using similar arguments as those in the proof of Lemma 3, we prove the following.

Lemma 7. *Let $\varepsilon_1, \varepsilon_2$ be two real numbers such that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Every planar graph with at least one edge has a $(\varepsilon_1, \varepsilon_2)$-Delaunay drawing.*

Proof. Again, the construction is along the lines of the proof of Lemma 3. In fact, the $(\varepsilon_1, \varepsilon_2)$-Gabriel drawing constructed in that proof *is* a $(\varepsilon_1, \varepsilon_2)$-Delaunay drawing: First note that, as the Gabriel disk of two points p and q is contained in the family of Delaunay disks of p and q, every edge in the drawing is correctly witnessed. To see that every non-edge is also correctly witnessed, note that in that proof of Lemma 3, we may choose C to be any disk with chord $\overline{pv_{i+1}}$ (i.e. any Delaunay disk) and its ε_2-expanded disk will contain D. □

The discussion of the section is summarized in the following theorem, which establishes that one can compute a drawing that approximates a Delaunay drawing for every planar graph, provided that the Delaunay disks are scaled down for the edges and are scaled up for the non-adjacent pairs of vertices by any arbitrarily small chosen amount.

Theorem 3. *Let G be an embedded planar graph with at least one edge. For any given values of $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, G admits an embedding preserving $(\varepsilon_1, \varepsilon_2)$-Delaunay drawing. Also, there exist embedded planar graphs that do not have an embedding preserving $(0, \varepsilon_2)$-Delaunay drawing and embedded planar graphs that do not have an embedding preserving $(\varepsilon_1, 0)$-Delaunay drawing.*

In all of these cases, restricting the approximation to the extreme cases that either $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$ prevents the drawing of all planar graphs. We now study the question of what subfamilies of planar graphs admit approximate proximity drawings in these extreme cases. We focus on $(0, \varepsilon_2)$-Gabriel drawings because they generalize the notion of weak Gabriel drawings.

5. $(0, \varepsilon_2)$-Gabriel drawings of some families of graphs

As mentioned in the introduction, $(0, \varepsilon_2)$-proximity drawings provide a unifying framework that encompasses both proximity drawings and weak proximity drawings. To give an impression of the families of planar graphs that can be represented using this framework, we focus in this section on families of graphs that have $(0, \varepsilon_2)$-Gabriel drawings. In view of the fact that not all maximal planar triangulations admit embedding preserving $(0, \varepsilon_2)$-Gabriel drawings (cf. Lemma 1), we consider families of planar graphs that are sparser than maximal planar triangulations. For the extremal cases of $(0, \infty)$-Gabriel drawings, that is, weak Gabriel drawings, and $(0, 0)$-Gabriel drawings the following is known: Di Battista et al. [8] proved that all biconnected outerplanar graphs and all trees have a $(0, \infty)$-Gabriel drawing, while Bose et al. [5] proved that not all trees have a $(0, 0)$-Gabriel drawing. The next two lemmas and Theorem 4 establish a tight threshold value for ε_2 for the $(0, \varepsilon_2)$-Gabriel drawability of embedded trees.

Lemma 8. *For any real number $\varepsilon_2 < 2$, there exists a tree that does not admit a $(0, \varepsilon_2)$-Gabriel drawing.*

Proof. Consider the star tree S_d with central vertex v of degree d. We show that if d is sufficiently large then S_d has no $(0, \varepsilon_2)$-Gabriel drawing for $0 \leq \varepsilon_2 < 2$.

Assume for a contradiction that Γ is a $(0, \varepsilon_2)$-Gabriel drawing of S_d. Select two distinct leaves u and w of S_d such that in Γ the angle α between $\overline{uv}$ and $\overline{vw}$ is minimal. By choosing $d \geq 9$, we have $\alpha < \pi/4$.

We assume without loss of generality that $d(v, w) \leq d(u, v) = 1$ holds. Let c denote the midpoint of $\overline{uw}$. The situation is depicted in Fig. 5. Note that, since $D(u, v)$ does not contain any vertices other than u and v, we have $d(v, w) \geq \cos \alpha$. This implies $\sin \alpha \leq d(u, w) \leq 2 \sin \frac{\alpha}{2}$.

Since α is minimal, the shaded area in Fig. 5, that is, the wedges with apex v and aperture angle α adjacent to the wedge defined by u, v and w, cannot contain any vertex in their interior. Hence, to obtain a lower bound on the minimum value by which $D(u, w)$ must be expanded to contain a vertex other than u and w, it

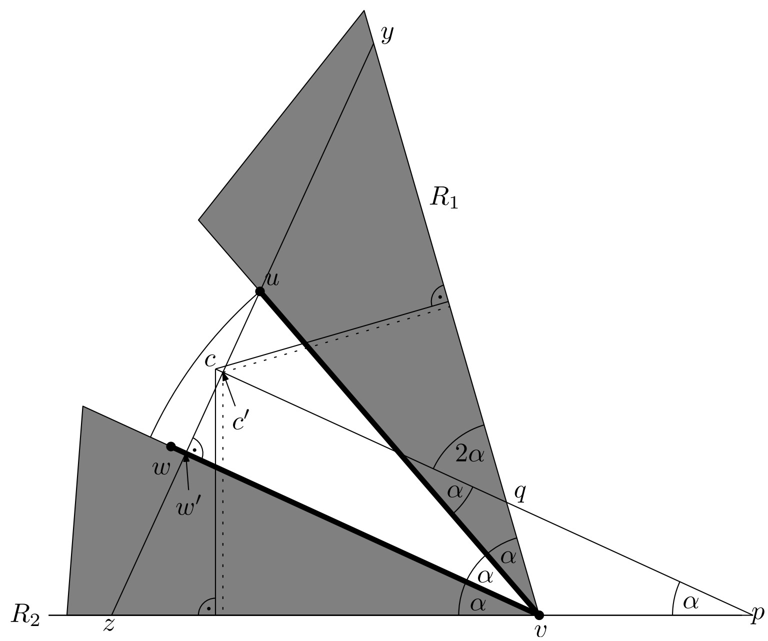

Figure 5: Proof of Lemma 8.

suffices to consider the minimum of $\frac{2d(c,v)}{d(u,w)}$, $\frac{2d_1}{d(u,w)}$, and $\frac{2d_2}{d(u,w)}$, where d_1 and d_2 denote the distance of c from the rays R_1 and R_2, respectively (see Fig. 5).

First note that, in view of $d(c, v) \geq d(w, v) \geq \cos\alpha$ and $d(u, w) \leq 2\sin\frac{\alpha}{2}$, we have $\frac{2d(c,v)}{d(u,w)} \geq \frac{\cos\alpha}{\sin(\alpha/2)}$ which tends to $+\infty$ as d tends to $+\infty$ and, thus, α tends to 0. Since triangles zpc' and zvw' are similar and $d(u, c') = d(c', w') = d(w', z)/2$, we have $d(c, p) \geq \frac{3}{2}\cos\alpha$. Since triangles $w'vy$ and $c'qy$ are similar and $d(u, y) \geq d(u, w')$, we have $d(c, q) \geq \frac{3}{4}\cos\alpha$. This implies, using again $d(u, w) \leq 2\sin\frac{\alpha}{2}$, that

$$\frac{2d_1}{d(u,w)} \geq \frac{d(c,q)\sin(2\alpha)}{\sin(\alpha/2)} \geq \frac{3\cos\alpha\sin(2\alpha)}{4\sin(\alpha/2)} \quad \text{and}$$
$$\frac{2d_2}{d(u,w)} \geq \frac{d(c,p)\sin\alpha}{\sin(\alpha/2)} \geq \frac{3\cos\alpha\sin\alpha}{2\sin(\alpha/2)}$$

hold. It is routine to check that the right hand sides in both inequalities above tend to 3 as d tends to $+\infty$. But this implies that, for sufficiently large d, the ε_2-expanded disk $D(u, w)$ for $0 \leq \varepsilon_2 < 2$ does not contain any vertices other than u and w, a contradiction. □

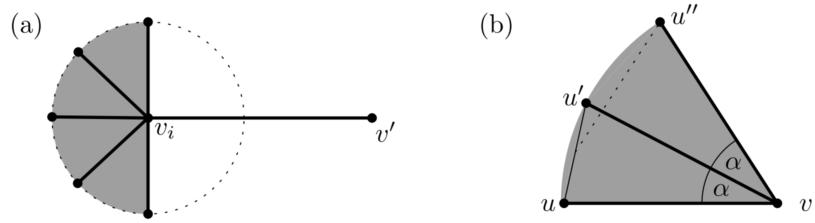

Figure 6: Proof of Lemma 9.

Lemma 9. *Let T be a tree. Then T admits a $(0, \varepsilon_2)$-Gabriel drawing for any real number $\varepsilon_2 \geq 2$.*

Proof. Select an arbitrary vertex v of T and fix an ordering $v = v_1, v_2, \ldots, v_n$ of the vertices of T such that for any $1 \leq i \leq j \leq n$ we have $d_T(v, v_i) \leq d_T(v, v_j)$ where $d_T(u, w)$ denotes the length of the path between vertices u and w in T. In addition, define, for all $1 \leq i \leq n$, $U_i := \{v_j : j > i$ and v_j is adjacent to $v_i\}$. To construct a $(0, \varepsilon_2)$-Gabriel drawing Γ of T, first place v at an arbitrary point in the plane. We now process the vertices $v_1, v_2, \ldots, v_n$ in that order. For each $1 \leq i \leq n$, the set W_i will contain those vertices of T that have already been assigned a position in the plane and the set E_i will contain those edges of T with both endpoints in W_i. We start with $W_1 = \{v_1\}$. When processing v_i this vertex is contained in W_i and after processing v_i all vertices in U_i, if any, also have been assigned a position and, thus, we have $W_{i+1} := W_i \cup U_i$.

To describe the construction of Γ, consider vertex v_i and assume all vertices v_j, $1 \leq j < i$, have already been processed. If $U_i = \emptyset$ then nothing needs to be done. So assume in the following that $U_i \neq \emptyset$. If $i \geq 2$ let v' denote the unique vertex adjacent to v_i not contained in U_i. If $i = 1$ define $r = 1$. Otherwise define r to be the minimum of $\frac{1}{2} \min\{d(v_i, w) : w \in W_i \setminus \{v_i\}\}$ and $\frac{1}{2} \min\{d(v_i, D(w, w')) : (w, w') \in E_i \setminus \{(v_i, v')\}\}$, where, for a point p and a disk D, $d(p, D)$ denotes the distance from p to the closest point in D. We place the vertices in U_i equidistantly on the boundary of the disk D_i centered at v_i with radius r as depicted in Fig. 6(a). In particular, all vertices are placed on the semicircle on the side opposite of v' using an ordering that maintains the given embedding. It follows immediately from the definition of r and the placement of the vertices in U_i that: (a) The disk $D(v_i, u)$ does not contain any vertices except v_i and u for all $u \in U_i$. (b) The ε_2-expanded disk $D(u, w)$ contains vertex v_i for all $u \in U_i$ and all $w \in W_i \setminus \{v_i\}$. (c) The edges (v_i, u), $u \in U_i$, do not cross any edge in E_i. (d) For every edge $(w, w') \in E_i$ the disk $D(w, w')$ does not contain any of the vertices in U_i.

It remains to show that the ε_2-expanded disk $D(u, u')$ contains some vertex in $W_{i+1} \setminus \{u, u'\}$ for any two distinct vertices $u, u' \in U_i$. This is immediately clear if the wedge defined by u, u' and v_i contains some vertex in $U_i \setminus \{u, u'\}$. So assume that u and u' are consecutive vertices on the boundary of D_i. Note that the angle α between $\overline{v_i u}$ and $\overline{v_i u'}$ equals $\frac{\pi}{\ell}$ for some integer $\ell \geq 2$. Now, it is easy to check that if α is at least $\pi/4$ then v_i is contained in the ε_2-expanded disk $D(u, u')$, and, if $\alpha \leq \pi/5$ holds, the vertex $u'' \in U_i$ that comes after u' is contained in the ε_2-expanded disk $D(u, u')$ (see Fig. 6(b)). □

Lemmas 8 and 9 can be summarized in the following.

Theorem 4. *Every tree has a $(0, \varepsilon_2)$-Gabriel drawing for any given value of ε_2 such that $\varepsilon_2 \geq 2$. Also, for each value of ε_2 such that $0 \leq \varepsilon_2 < 2$, there exists a tree T such that T does not have a $(0, \varepsilon_2)$-Gabriel drawing.*

We now consider outerplanar graphs with cycles. Lenhart and Liotta [15] proved that all biconnected outerplanar graphs with a given outerplanar embedding have a $(0, 0)$-Gabriel drawing that maintains the embedding, while a connected outerplanar graph where a cut vertex is shared by more than four biconnected components is not $(0, 0)$-Gabriel drawable. The next theorem shows that this upper bound on the number of components sharing a cut vertex can be removed in $(0, \varepsilon_2)$-Gabriel drawings, provided that the input graph does not have degree one vertices. In the statement, by embedded outerplanar graph we mean an outerplanar graph with a planar embedding where all vertices are on the external face.

Theorem 5. *Let G be an embedded outerplanar graph that does not have vertices of degree one or zero. G has a $(0, \varepsilon_2)$-Gabriel drawing that maintains the embedding for any given value of ε_2 such that $\varepsilon_2 > 0$.*

Proof. Let G be a connected outerplanar graph where each vertex has degree > 1. Let $\varepsilon_2 > 0$. At the end of the proof we will show that the theorem also holds for disconnected outerplanar graphs. We call the edges of G the blue edges. We first add new edges to G to create a new graph G' that is outerplanar, internally triangulated and biconnected. The new edges are called red edges. A vertex v of G' will be drawn at a point in the drawing Γ that we will also denote by v. Let $(1 + \varepsilon_2)D(u, v)$ be the ε_2-expanded version of the Gabriel disk $D(u, v)$.

Let (a, b) be an arbitrary edge on the outerface of G'. Let c be the third point of the triangle containing (a, b). We first draw triangle (a, b, c) followed by all other triangles of G' in the following order. Since G' is outerplanar, internally triangulated and biconnected, the dual of G' is a binary tree T. We execute any top-to-bottom traversal of T, starting at the vertex corresponding to triangle (a, b, c), for example a breadth first traversal. We draw the triangles of G' in the order that they are visited in the traversal of T. Suppose vertex t is a child of vertex s in T. If t corresponds to triangle (v, w, x) and s corresponds to triangle (u, v, w), we say that vertex x is a child of the edge (v, w).

We place the two points a and b at distinct locations in the plane. We place c outside $D(a, b)$, but inside $(1 + \varepsilon_2)D(a, b)$. The remainder of the points will be placed in two regions as illustrated in Fig. 7. Region c_l lies outside $D(a, c)$,

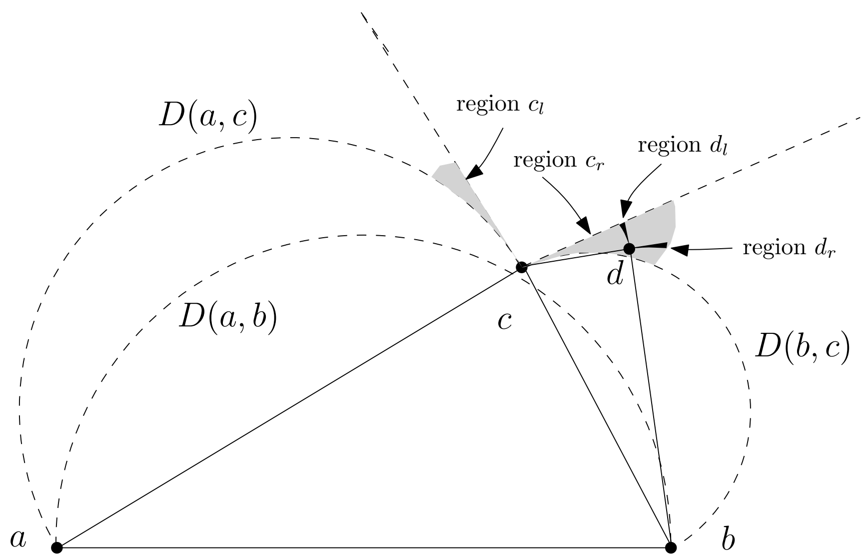

Figure 7: $(0, \varepsilon_2)$-Gabriel drawing of a biconnected outerplanar graph.

outside the triangle (a, b, c), but inside $(1 + \varepsilon_2)D(b, c)$. Moreover any point p in region c_l forms an angle $\angle(p, c, a)$ that is less than 90 degrees. Region c_r lies outside $D(b, c)$, outside the triangle (a, b, c), but inside $(1 + \varepsilon_2)D(a, c)$. Moreover any point p in region c_r forms an angle $\angle(p, c, b)$ that is less than 90 degrees. The vertices of the subgraph of G' adjacent to (a, c) will be placed in c_l, the vertices of the subgraph of G' adjacent to (b, c) will be placed in c_r. Suppose there is a triangle (b, c, d) in G'. We place d in region c_r, and recursively construct two regions d_l and d_r for the points of G' that are in the subgraph adjacent to (c, d) and in the subgraph adjacent to (b, d) respectively. Continuing to process G' according to the chosen traversal of T yields a $(0, \varepsilon_2)$-Gabriel drawing of G'.

We now have to remove the red edges from the drawing. First of all, any edge (u, v) not on the boundary of G' has a child vertex w that is placed outside $D(u, v)$ and inside $(1 + \varepsilon_2)D(u, v)$. So any edge not on the boundary of G' can be removed. The same argument can be used to remove the edge (a, b) if it is red. Now let (u, v) be a red edge on the boundary of G' not equal to (a, b). Let u and v be vertices of the triangle (u, v, w). We can assume without loss of generality that u is the child of (v, w). Since u has degree > 1 in G, the edge (u, w) has a child x in G', so there is an edge (u, x) in G' such that x is placed in region u_l or region u_r. Since this region falls inside $(1 + \varepsilon_2)D(u, v)$, we can delete the edge (u, v).

Finally assume that G is not connected. We first draw the connected components as explained above. We place the components in the plane sufficiently far away from each other. Since we do not have components of size 1 we can conclude that for any two vertices u and v in different components, there are vertices inside $(1 + \varepsilon_2)D(u, v)$. □

6. Conclusions and open problems

In this paper we have introduced an approximate version of several well-studied proximity drawings. In comparison with the standard definition of region of influence based proximity drawing, our drawings consider a slightly smaller region of influence for the adjacent pairs of vertices and a slightly larger region for the non-adjacent pairs. The amount by which the region of influence can be scaled up or down depends on two non-negative real numbers ε_1 and ε_2; the resulting straight-line drawing is called an $(\varepsilon_1, \varepsilon_2)$-proximity drawing. Intuitively, the smaller these parameters are the closer an $(\varepsilon_1, \varepsilon_2)$-proximity drawing is to the standard proximity drawing.

We investigated the approximation of three well-known proximity drawings, namely Gabriel drawings, β-drawings, and Delaunay drawings. For each of these types of proximity drawings, we showed that every planar graph has a planar straight-line drawing that can be made arbitrarily close to satisfying the usual proximity rule. This contrasts with well-known results that only restricted subfamilies of planar graphs have a (standard) Gabriel drawing, or β-drawing, or Delaunay drawing. We also investigated extremal cases that generalize and extend the notion of weak proximity. A first natural direction for future research is therefore the following.

Question 1. *Extend the study of approximate proximity to other classical or emerging families of proximity drawings, such as rectangle of influence drawings [10], witness rectangle graphs [3], and witness Delaunay drawings [2].*

We remark that the major contribution of this paper is in analyzing to what extent the class of representable graphs can vary if the standard definition of proximity is approximate in the manner described above. Based on the presented results, we believe that the proposed definition of approximate proximity may be effectively adopted in practice to represent planar graphs where proximity constraints need to be maintained. However, in order to do so, relevant questions about the area and the bit complexity of the computed drawings must be addressed.

As an example, we recall a recent paper by Angelini et al. [1] proving that exponential area is required to draw some trees of maximum degree five as Euclidean minimum spanning trees. Since the family of β-drawable trees for $\beta = 2$ is the family of trees having maximum degree five and since a 2-drawing of a tree is also a Euclidean minimum spanning tree, it follows that exponential area is required to draw some trees as $(0, 0)$-2-drawings. On the other hand, every straight-line planar drawing is an $(\varepsilon_1, \varepsilon_2)$-proximity drawing for sufficiently large values of ε_1 and ε_2. In fact, every planar graph has a $(0, \infty)$-proximity drawing with integer coordinates using polynomial area [6]. This discussion leads to the following research direction.

Question 2. *Study polynomial area approximation schemes, where the size of the computed drawing increases as the variables ε_1, ε_2 decrease. Similar studies have been done in the context of drawing a tree as a Euclidean minimum spanning tree [13] and for rectangle of influence drawings [10].*

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