3. Let $B(X)$ be the best woman that man X is paired with in some stable pairing. We want to show that the proposal algorithm produces the pairing $(X, B(X))$ for all men X. (Note: At this point we don't even know that the $B(X)$ are distinct. For all we know, $B(X)$ may equal $B(Y)$ for $X \neq Y$.) Let $G(X)$ be the woman that is paired with man X by the Gale-Shapley proposal algorithm. Suppose $G(X) \neq B(X)$. Woman $G(X)$ is less preferable to X than woman $B(X)$, thus $B(X)$ rejected X during the proposal algorithm. Suppose man X is the first man rejected by his optimal stable woman during the execution of the proposal algorithm. Let Y be the man that $B(X)$ prefers to X at the time of X's rejection. Man Y has not been rejected by his optimal stable partner, $B(Y)$, so he prefers $B(X)$ to $B(Y)$ or $B(Y) = B(X)$. Now consider a stable pairing P that contains $(X, B(X))$. We know $B(X)$ prefers Y to X, and Y prefers $B(X)$ to his partner in P. (Y prefers $B(X)$ to *any* stable partner not equal to $B(X)$.) Thus any pairing that contains $(X, B(X))$ is not stable, contradicting our definition of $B(X)$ as X's optimal stable partner.

5. Run the proposal algorithm with the men doing the proposing. The algorithm will terminate once m different women receive a proposal, since at that point all the men are paired. (Note that this uses the fact that once a woman is paired, she remains paired during the proposal algorithm.) Since $m < n$, no man runs out of women to propose to before this occurs and the stability of the resulting pairing follows as discussed in class.

   Let G be the resulting pairing. Suppose there exists another stable pairing P such that a woman y is unpaired in P and paired in G to some man Y. By a similar argument to that given in question 3, man Y prefers woman y to any other woman he is paired with in a stable pairing. Thus man Y prefers woman y to his partner in P and woman y prefers man Y to being unpaired, so pairing P is unstable.

6. (from a solution by David Zuckerman) The game ends when the last king is drawn. To find the expected number of cards remaining, consider a circle with 53 items on its perimeter. We select 5 different ones uniformly at random – 4 of these represent the 4 kings and 1 represents where we break open the circle to form a linear ordering of the remaining 52 items. Let X_1 be a random variable describing the distance from the splitting item to the first king (clockwise), X_i denote the distance from the $(i-1)$st king to the ith king ($i = 2, 3, 4$), and X_5 the distance from the fourth king to the splitting item. By symmetry, $E[X_i] = E[X_j]$. Moreover, $X_1 + X_2 + X_3 + X_4 + X_5 = 53$. So $E[X_i] = 53/5$ for $i = 1, 2, 3, 4, 5$. The expected number of cards remaining on the table is $E[X_5] - 1 = 48/5$.