CS 500 | Problem Set 1 | 9 Jan 2012
W. Evans | Due: 16 Jan 2012

Optional background reading from Randomized Algorithms, Motwani, Raghavan (MR).
The Stable Marriage Problem pages 53–59

You may discuss problems with other people in the class, but you must write up your own solutions. If you do discuss a problem with someone else or you use an outside resource, you must acknowledge them. Do not copy solutions from anyone. A star ($\star$) means that the problem may be somewhat difficult.

1. We gave an iterative process that starts with an arbitrary pairing and repeatedly modifies it by placating a dissatisfied couple (i.e. letting them run off together and pairing up their former partners). Give an example of three couples in which this process cycles for some initial pairing. You should provide the preference lists for the three men and three women, and the cycle of pairings.

2. Prove that the worst case number of proposals made by the Gale-Shapley proposal algorithm is $n^2 - n + 1$. This requires you to show that no more proposals are ever made **and** (perhaps harder) to describe an example, for general n, that requires this many proposals.

3. $\star$ Let $G(X)$ be the woman that is paired with man X by the Gale-Shapley proposal algorithm. Prove that there is *no* stable pairing that pairs man X to a woman he prefers more than $G(X)$.

4. Consider a monosexual society with $2n$ individuals, each of whom has a linearly ordered preference list of the other $2n - 1$ individuals. A matching of these individuals into pairs is called *stable* if it does not contain two pairs, XY and ZW, such that X prefers Z to Y and Z prefers X to W. Give an example in which no stable pairing exists. An example with four individuals exists.

5. Suppose the number m of men is different than the number n of women. A pairing P is *unstable* in this case if there is a man A and a woman b such that

   (a) A and b are not paired in P, and

   (b) A is either unpaired in P or prefers b to his partner in P, and

   (c) b is either unpaired in P or prefers A to her partner in P.

   Prove that if $m < n$ there is at least one stable pairing in which all the men are matched. (This should be easy using the Gale-Shapley algorithm.)

   Prove that if $m < n$ there is a set of $n - m$ women who are never matched in any stable pairing.

6. $\star$ What is the expected number of cards left on the table when the clock solitaire game (discussed in class) terminates?