Note Title

Rigid Body Dynamics

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state

$p \in \mathbb{R}^3$ position
$v \in \mathbb{R}^3$ velocity
] particle

$p \in \mathbb{R}^3$ position of COM
$v \in \mathbb{R}^3$ velocity of COM
$R \in \mathbb{R}^9$ angular position, i.e. orientation
3x3 rotation matrix
$w \in \mathbb{R}^3$ angular velocity
] rigid body

Equations of Motion

$$\dot{a} \equiv \frac{da}{dt}$$

$$\dot{p} = v$$
$$\Sigma F = m\dot{v}$$
] particles

$$\dot{R} = ?$$
$$\dot{w} = ?$$
$$\dot{p} = v$$
$$\Sigma F = m\dot{v}$$
] rigid bodies

Integrate

$$p = p + \dot{p}\Delta t$$
$$v = v + \dot{v}\Delta t$$
$$R = R + \dot{R}\Delta t$$
$$w = w + \dot{w}\Delta t$$
] rigid bodies

Definitions & Building Blocks

$$\vec{a} \times \vec{b} = \tilde{a}\,\vec{b}$$

$$[\ ] \times [\ ] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$L = r \times r \times w = \underbrace{\tilde{r}\,\tilde{r}}\,w$$

$$\vec{w} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$ angular velocity

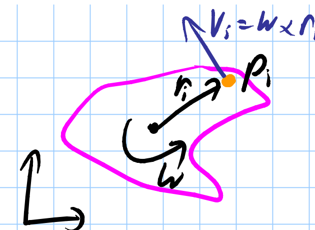
$|w|$: how fast the object is spinning

$\frac{w}{|w|}$: axis that the object is spinning about

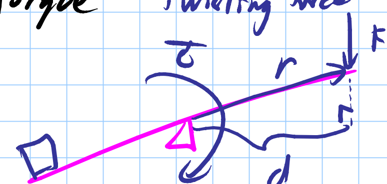
$$w = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}$$

RH rule: thumb ⟷ w
finger curl gives direction of spin

$$\boxed{v_i = w \times r_i}$$



torque "twisting force"



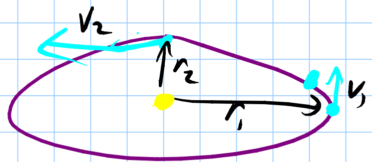
$$\tau = F \cdot d$$
$$\boxed{\tau = r \times F}$$

linear momentum $\vec{P} = m\vec{v}$

angular momentum $\tau = r \times F$

$$\boxed{L = r \times P}$$

example



$$L_1 = r_1 \times (m v_1)$$

$$|L_1| = r_1 \cdot m \cdot v_1 = r_2 \cdot m \cdot v_2$$

Eq'ns of Motion

$$\sum F = \frac{dp}{dt} = \frac{d(mv)}{dt} = \cancel{\dot{m} v} + m \dot{v} = m \cdot a$$

$$p = m v$$

$$L = I w$$

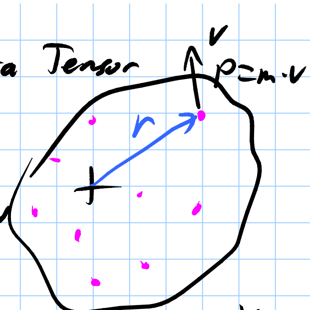
$$\sum \tau = \frac{dL}{dt} = \frac{d(I w)}{dt} = \dot{I} w + I \dot{w}$$

$$\boxed{\sum \tau = w \times I w + I \dot{w}}$$

Derivation of Inertia Tensor

$$L = r \times p$$

for a set of particles



$$V_i = w \times r$$

$$L = \sum_i r_i \times p_i$$

$$= \sum_i r_i \times (m_i v_i)$$

$$= \sum_i m_i (r_i \times w \times r_i)$$

$$= \sum_i m_i \tilde{r}_i (w \times r_i)$$

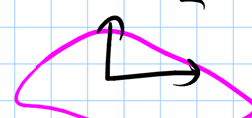
$$= \underbrace{\boxed{\sum_i - m_i \tilde{r}_i \tilde{r}_i}}_{I} w$$

inertia tensor

$$L = \sum_i - m_i \tilde{r}_i \tilde{r}_i \vec{w}$$

$$= \sum_i - m_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \vec{w}$$

$$= \begin{bmatrix} \sum_i m_i (z_i^2 + y_i^2) & \sum_i - m_i x_i y_i & \sum_i - m_i x_i z_i \\ & & \\ & & \sum_i m_i (x_i^2 + y_i^2) \end{bmatrix} \vec{w}$$



$$= I \vec{w}$$

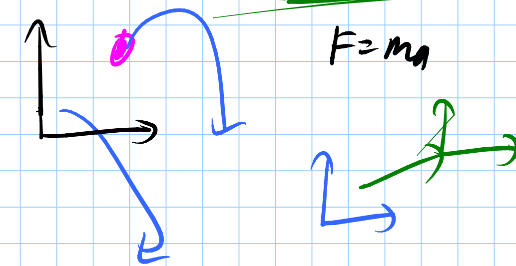
$$\sum_i m_i (z_i^2 + y_i^2) \longrightarrow \int_m dm (z^2 + y^2)$$

$\rightarrow \int_V \rho \, dV (z^2+y^2)$

$\rightarrow \int_x \int_y \int_z \rho (z^2+y^2) \; dz\, dy\, dx$

$$I_{block} = \begin{bmatrix} I_{xx} & & 0 \\ & I_{yy} & \\ 0 & & I_{zz} \end{bmatrix}$$

But: eqn's of motion only hold in an inertial frame



$F = ma$

$$\Sigma\tau = \omega \times I\omega + I\dot{\omega}$$

$L = I\omega$ $\qquad I_B \rightarrow I \qquad I = R I_B R^T$

$L_B = I_B \omega_B$ $\qquad L = R I_B \omega_B$

$L = R L_B$ $\qquad L = \boxed{R I_B R^T}\omega$

$\omega = R\omega_B \rightarrow \omega_B = R^{-1}\omega$ $\qquad L = I\omega$

for each timestep
- compute external forces & torques
- compute $I = R I_B R^T$
- eqns of motion

$$\left.\begin{aligned} \Sigma F &= m\dot{v} \\ \Sigma\tau &= \omega \times I\omega + I\dot{\omega} \end{aligned}\right] \text{Newton-Euler eqns of motion}$$

- integrate

$$\left.\begin{aligned} p &= p + v\Delta t \\ v &= v + \dot{v}\Delta t \\ R &= R + \dot{R}\Delta t \\ \omega &= \omega + \dot{\omega}\Delta t \end{aligned}\right] \text{forward Euler integration}$$

$$\dot{R} = \tilde{\omega} R$$

end

$$\begin{bmatrix} m & & & \\ & m & & \\ & & m & \\ & & & I \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \Sigma F \\ \Sigma\tau - \omega \times I\omega \end{bmatrix}$$