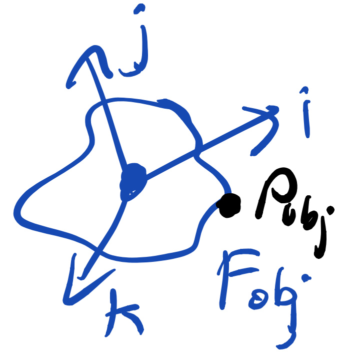
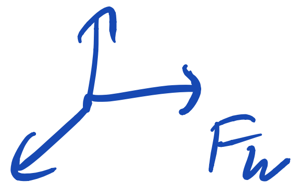
Animating an Object

Transformation Matrix

$P_w = M P_{obj}$



$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_w = \begin{bmatrix} i & j & k & O_{obj} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{obj}$$



$$\begin{bmatrix} R & T \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

for rigid body transformation
i.e., no scaling, no deformation

Representing Orientations

Are these rotation matrices ?

SO(3) special ortho~~gonal~~ normal

yes $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rot(z, 90°)

no $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

$|i|, |j|, |k| \neq 1$

no $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$x' = -x$ mirrors object around x (through yz plane)

$\underset{3\times 3}{R}$ $\begin{bmatrix} a & b & c \end{bmatrix}$

numbers: 9

constraints:

$|a| = 1$ $a \cdot b = 0$ $\det(R) = +1$

$|b| = 1$ $a \cdot c = 0$

$|c| = 1$ $b \cdot c = 0$

degrees of freedom $= 9 - 6 = 3$

Representing Rotations

	numbers	constraints	
1. 3x3 Rotation matrix	9	6	
2. Euler Angles	3	0	
3. Angle-Axis	4	1	$\lvert u \rvert = 1$
(exponential map)	3	0	
4. Quaternions			

Euler Angles

3 successive rotations about changing axes

Let's call the angles α, β, γ: → rotations wrt moving frame.

$XYZ: Rot(x,\alpha)\, Rot(y,\beta)\, Rot(z,\gamma)$

$ZYX: Rot(z,\alpha)\, Rot(y,\beta)\, Rot(x,\gamma)$

$XYX: Rot(x,\alpha)\, Rot(y,\beta)\, Rot(x,\gamma)$

# of possible Euler angle spec:

3 choices $\times$ 2 choices $\times$ 2 choices $= 12$

issues: - non-unique $ZYX(0,90,0) = ZYX(90,90,90)$

- gimbal lock: $XYX(20,0,30)$

$A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ no rotation

$A = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}$ Rot(x, 180)

Angle-Axis

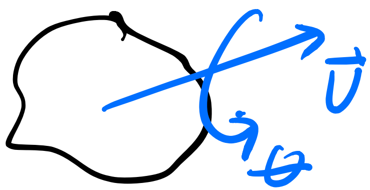
- Euler's Rotation Theorem (1776)

  Theorema. Quomodocunque sphaera circa centrum suum conuertatur, semper assignari potest diameter, cuius directio in situ translato conueniat cum situ initiali.

  When a sphere is moved around its centre it is always possible to find a diameter whose direction in the displaced position is the same as in the initial position.

→ can move btwn any two orientations with a single rotation about some axis.

$$R = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$



Rotate($\vec{U}$, θ) : U_x U_y U_z θ

- most graphics APIs implement this.
- can also use $\vec{A} = \theta\vec{U}$ $|\vec{A}| = \theta$

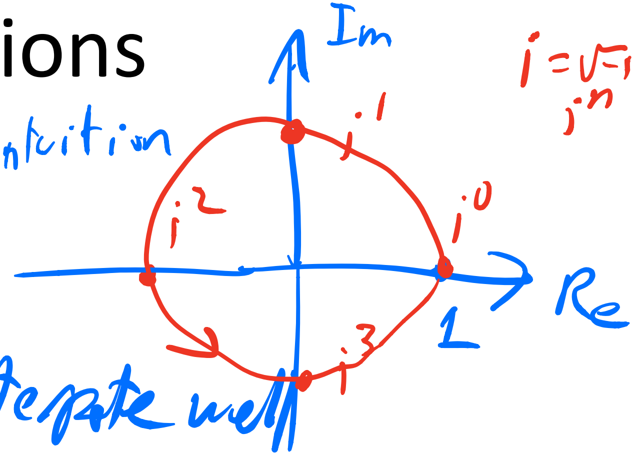
$\vec{U} = \vec{A}/|A|$

$\boxed{a + bi}$

$\cancel{a} + \boxed{bi + cj + dk}$

Unit Quaternions

Intuition



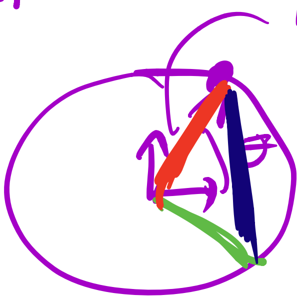
$i = \sqrt{-1}$

i^n

- desiderata
  - compact
  - unique
  - continuous, i.e. should interpolate well
  - easy to compose

unit circle

$r = 1$



$\theta \in [0, 2\pi)$

r_x, r_y $\quad |r| = 1$

$\cos\theta, \sin\theta$

sphere

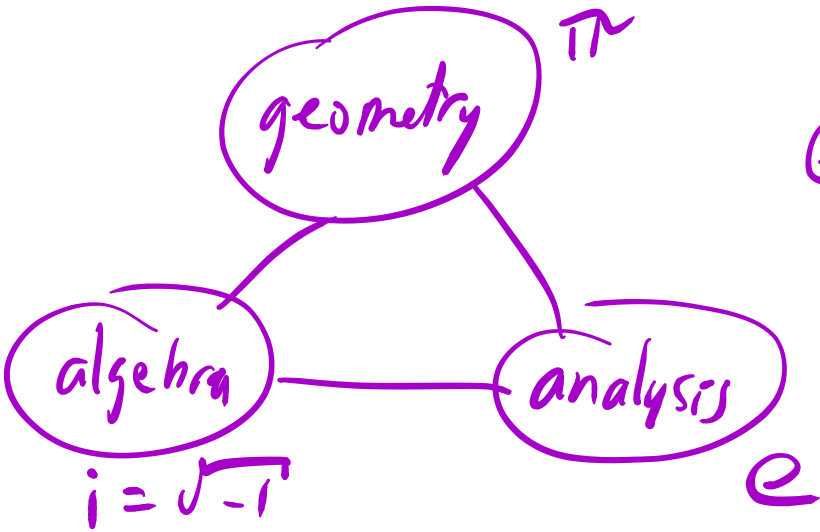
θ, ϕ

r_x, r_y, r_z

$|r| = 1$

A bit more on Euler ….

- Notations introduced: $\pi, i, e, f()$
- Historical context



$$e^{i\pi} + 1 = 0$$

"most beautiful eqn"

Euler and Complex Exponentials

$e^x \triangleq$ a function that equals its own derivative

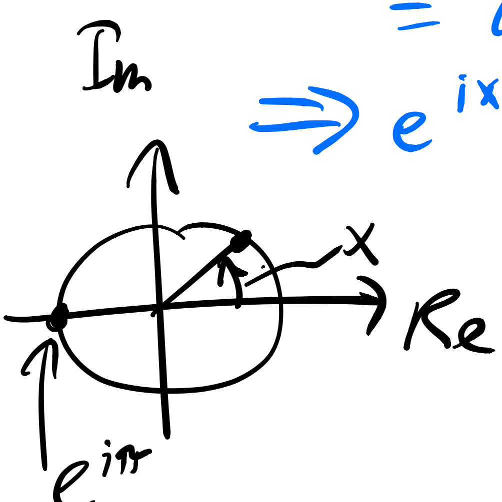
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos(x) + i\sin(x)$$

$\Rightarrow e^{ix}$ can represent a rotation in the complex plane



$x = \pi \quad e^{i\pi} + 1 = 0$

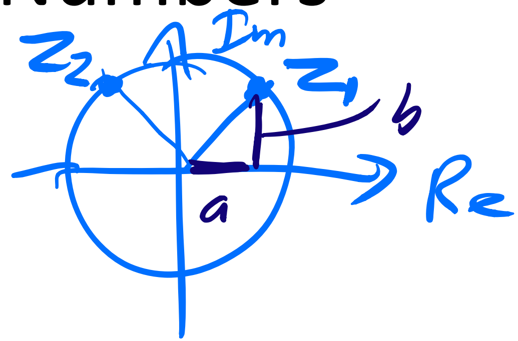
"most beautiful eqn"

Multiplying Complex Numbers

Complex numbers

$i^2 = -1$

$z = a + bi$



$$\left.\begin{aligned} a_1 &= \cos\theta_1 \\ b_1 &= \sin\theta_1 \end{aligned}\right\} \quad \begin{aligned} z_1 &= e^{i\theta_1} \\ z_1 &= a_1 + b_1 i \end{aligned}$$

$$\left.\begin{aligned} a_2 &= \cos\theta_2 \\ b_2 &= \sin\theta_2 \end{aligned}\right\} \quad \begin{aligned} z_2 &= e^{i\theta_2} \\ z_2 &= a_2 + b_2 i \end{aligned}$$

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1) \\ &= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \end{aligned}$$

or

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

$\Rightarrow$ multiplying unit complex numbers adds their angles in the complex plane.

Beyond complex numbers…

quaternions.

- commutative

$z_1 \cdot z_2 = z_2 z_1$

- associative

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

- distributive

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

z	a	$a+bi$	$a+bi+cj$	$a+bi+cj+dk$	
commutative	✓	✓	✗	✗	✗
associative	✓	✓	✗	✓	✗
distributive	✓	✓	✗	✓	✓
	1D	2D	3D	4D	8D

Quaternion Definitions

- form $q = w + xi + yj + zk$

  $= (w, x, y, z)$

  $= (s, v)$ where $s = w$, $v = (x, y, z)$

"normal"

- unit quaternion $\|q\| = 1$ $\quad x^2 + y^2 + z^2 + w^2 = 1$
- addition $q_1 + q_2 = (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2)$
- multiplication $i^2 = j^2 = k^2 = -1$

  $i \cdot j = k \quad j \cdot i = -k$

  $j \cdot k = i \quad k \cdot j = -i$

  $k \cdot i = j \quad i \cdot k = -j$

- Conjugate

$\bar{q}(s, v) = (s, -v)$

Quaternion Multiplication

$$q_1 \otimes q_2 = (w_1 + x_1 i + y_1 j + z_1 k)(w_2 + x_2 i + y_2 j + z_2 k)$$

	w_2	$x_2 i$	$y_2 j$	$z_2 k$
w_1	$w_1 w_2$	$w_1 x_2 i$	$w_1 y_2 j$	$w_1 z_2 k$
$x_1 i$	$w_2 x_1 i$	$-x_1 x_2$	$x_1 y_2 k$	$-x_1 z_2 j$
$y_1 j$	$w_2 y_1 j$	$-x_2 y_1 k$	$-y_1 y_2$	$y_1 z_2 i$
$z_1 k$	$w_2 z_1 k$	$x_2 z_1 j$	$-y_2 z_1 i$	$-z_1 z_2$

$$(s_1, v_1) \otimes (s_2, v_2) = (s_1 s_2 - v_1 \cdot v_2,\ s_1 v_2 + s_2 v_1 + v_1 \times v_2)$$

$$q_1 \otimes q_2 = (\underset{4}{\text{real terms}},\ \underset{4}{i \text{ terms}},\ \underset{4}{j \text{ terms}},\ \underset{4}{k \text{ terms}})$$

Unit Quaternions

$$\hat{k} = \frac{\vec{k}}{\|\vec{k}\|}$$

- angle-axis equivalent

$$Rot(\vec{k}, \theta) = \left(\overbrace{\cos\left(\frac{\theta}{2}\right)}^{s}, \overbrace{\sin\left(\frac{\theta}{2}\right)\hat{k}}^{v}\right)$$

$$= (w, \langle x, y, z \rangle)$$

- composition

with rotation matrices $R = R_1 R_2$

$$q = q_1 \otimes q_2$$

$$(q_1 \otimes q_2) \otimes q_3 = q_1 \otimes (q_2 \otimes q_3)$$

"Unit" Quaternion rotation of a point

$p' = Rot(\vec{k}, \theta)\, p$

$\Downarrow$

q

$\tilde{p}' = q \otimes \tilde{p} \otimes \bar{q}$

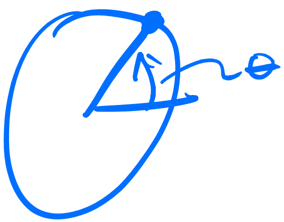
$\tilde{p} = (0, p)$

$\bar{q} = (s, -v)$

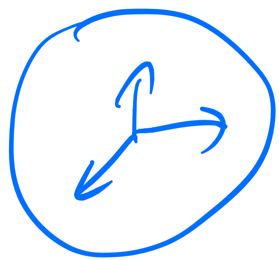
Why $\theta/2$?

Circle

$\cos\theta, \sin\theta$



Sphere

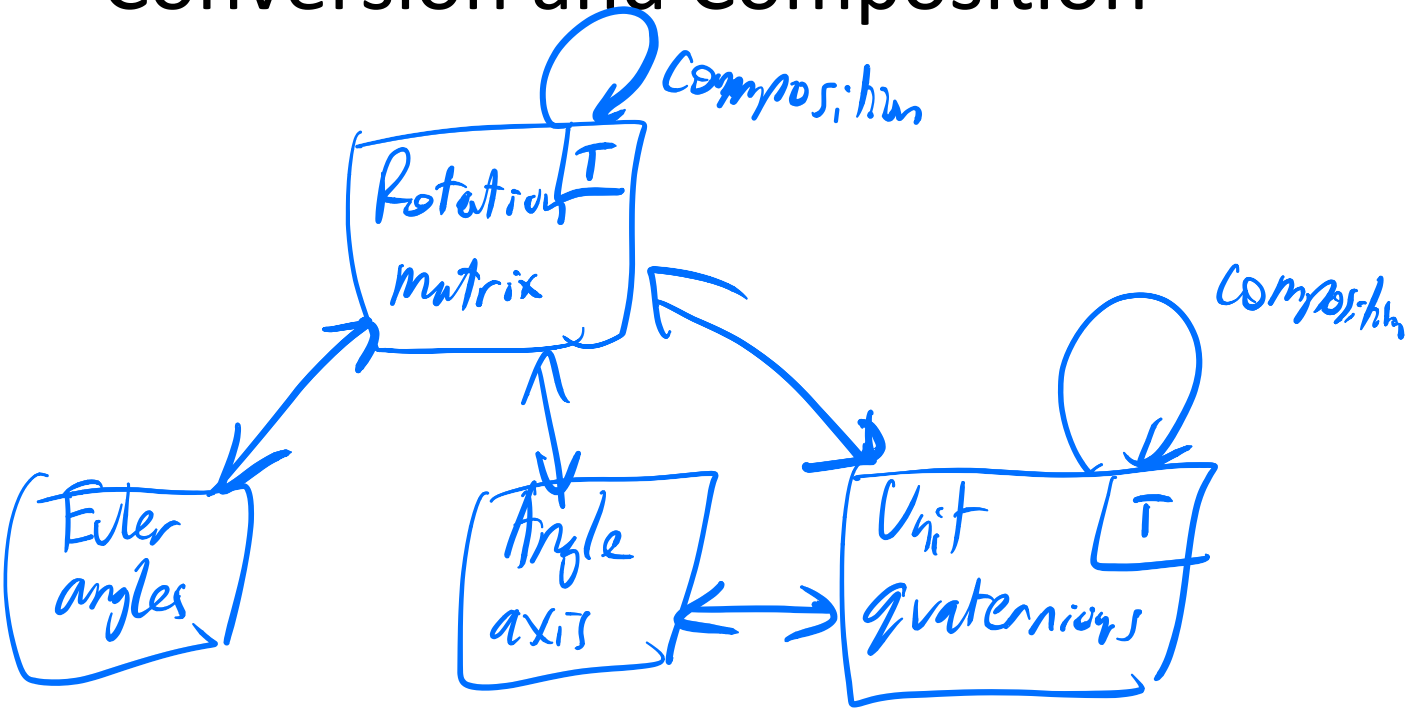


Comments

Summary

	Compact	easy to interpolate	unique	easy to compose
• Rotation matrices	✗	✓ (**)	✓	✓
• Euler angles	✓	✗	✗	✗
• Angle-axis	✓	?	✓ (*)	✗
• Unit Quaternions	✓	✓ (*)	✓ (*)	✓

Conversion and Composition



T = can be used directly to transform pts

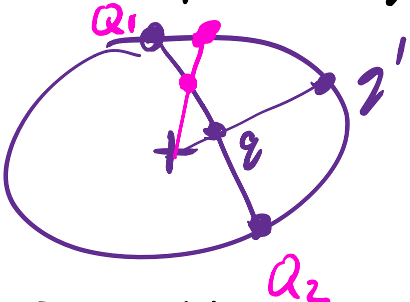
Interpolation of Unit Quaternions

Idea: interpolate the components

$t \in [0,1] \quad Q(t) = (1-t)\,Q_1 + t\,Q_2$

Issues:

① Interpolated quaternions need to be renormalized.



– reproject to unit length.

② Interpolate from

$Q_1 \rightarrow Q_2$

$Q_1 \rightarrow -Q_2$

$-Q_1 \rightarrow Q_2$

$-Q_1 \rightarrow -Q_2$

two choices

Compute $\cos\theta_A = Q_1 \cdot Q_2$

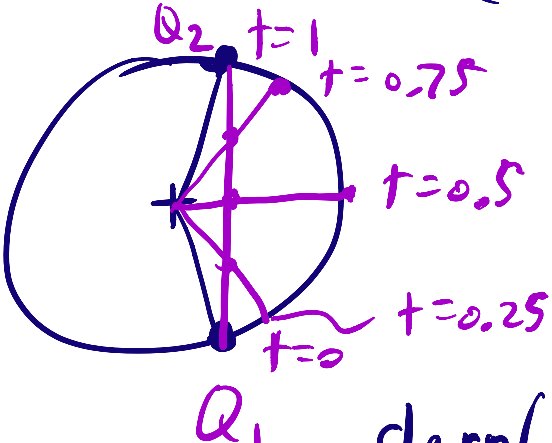
$\cos\theta_B = Q_1 \cdot (-Q_2)$

then take shortest path

SLERP

spherical linear interpolation

$$Q(t) = (1-t)Q_1 + tQ_2 \qquad t \in [0,1]$$



Issue: evenly-spaced interpolation $\neq$ evenly-spaced orientations.

$$\text{slerp}(Q_1, Q_2, t) = \frac{\sin((1-t)\theta)}{\sin\theta} Q_1 + \frac{\sin(t\theta)}{\sin\theta} Q_2$$

where $\cos\theta = Q_1 \cdot Q_2$

Interpolating Rotation Matrices

- Linear interpolation

e.g. $0.5\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} + 0.5\begin{bmatrix}0 & -1 & \\ 1 & 0 & \\ & & 1\end{bmatrix} = \begin{bmatrix}0.5 & -0.5 & \\ 0.5 & 0.5 & \\ & & 1\end{bmatrix}$

no rotation

$Rot(z, 90°)$

$\dfrac{Rot(z, 45°)}{\sqrt{2}}$

Object shrinks!

- Matrix exponential

$M = Rot(z, 90°)$

$M^2 = Rot(z, 90°)\,Rot(z, 90°) = Rot(z, 180°)$

$M^{0.5} \stackrel{?}{=} Rot(z, 45°)$

Can work, but not often used.

$\underset{t=0}{R_1} \rightarrow \underset{t=1}{R_2} \implies R_1 (R_1^{-1} R_2)^t$