High Dimensionality I

Manifold Methods

Talk Overview

• Define Concepts and Problems
• Paper 1: Charting A Manifold by Matthew Brand
• Paper 2: Maximum Likelihood Estimation of Intrinsic Dimension by Elizaveta Levina and Peter J. Bickel
• Discussion

Common Scientific Problem

• Make N observations

• Make a series of M measurements per observation

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• Make N observations

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• NOW WHAT?

Visualization

• Directly Visualize Dimensions
  – Parallel Coordinates
  – Glyphs
  – Star Coordinates
  – Etc.

Problem: Hidden Factors

True Dimensionality < Measured Dimensionality
Example

- Rotating head
- Large Number of Measured Dimensions
- Low Number of “Intrinsic” Dimensions

Solution: Dimensionality Reduction

- Find the true dimensionality
- PCA – Find Largest Axes of Variability And Construct a Plane
- MDS – Embed points based on Distances

Problem

What is a Manifold?

- A topological space that looks locally like the Euclidean space $\mathbb{R}^n$
What is a Manifold?

So What’s the Problem?

Manifold

So What’s the Problem?

PCA

So What’s the Problem?

PCA
So What’s the Problem?

MDS

Euclidean Distance

“Classic” Manifold Method: ISOMAP

1. Link To Nearest Neighbors

2. Compute Distances THROUGH Graph

“Real” Distance

“Classic” Manifold Method: ISOMAP

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Paper I: Charting a Manifold
Matthew Brand

Why Bother?

What’s Going On?

- Isomap depends on the integrity of the local structure of the manifold
- Noise perturbs the structure leading to an incorrect embedding.
What do we do about it?

What if we merged the similar frames?

And weighted them according to our confidence?

Gaussian Mixture Model
Estimates Multimodal Sample Density

Derive Coordinate Frame From Eigenvectors Of Distribution
Gaussian Mixture Model
These “Soft” Regions Are our Charts

Semi-Invertable Transform
• A transformation to and from the manifold
  - Three principal degrees of freedom captured from raw images:
    - pose
    - scale
    - expression
  - Interpolate on the manifold and “backproject” to original sample space

Critique
• GOOD
  - Elegant, robust idea solves shortcomings of former methods
  - Lots of novel examples to prove utility
  - Backprojection provides visualization opportunities
• BAD
  - Little appeal to intuition
  - No Code
  - Runtimes? How does it scale?

Paper II: Maximum Likelihood Estimation of Intrinsic Dimension
Elizaveta Levina and Peter J. Bickel

How many eigenvectors?
We use only the largest D where
D = Intrinsic Dimensionality
How do we get D?

• Most often = User makes a guess
• Use an estimation method
  – Projection Methods (PCA, local PCA)
  – Geometric Methods

Geometric Methods

• $C(r)$ = average number of points in radius $r$ for each point in dataset
• Plot $\log(C(r))$ against $\log(r)$
• $D = \text{slope}$

Why?
Why? C(r) grows like x

Log(x)/\log(x) = 1
C(r) grows like $x^2$

Log($x^2$)/log(x) = 2

This is called the correlation dimension

How well does this work?

Issues

• We don’t know the effect of
  – Sample Size
  – Dimension

• We also don’t understand bias or variance

Strategy of Paper II

• Define a stochastic process to model observations in sphere for some low dimensional density.

• Define a MLE for the dimension parameter of the process.

• Examine statistical properties of the estimator.

Step 1: Define the Process

• $N(t,x) =$ number of points in a sphere of radius t around point x

• We approximate this with a Poisson process

• The rate of this process depends on D!

Step 2: Define the MLE

• MLEs infer values of parameters of underlying process.

• Build an MLE for D

\[
\left[ \frac{1}{k-1} \sum_{j=1}^{k-1} \log \frac{T_k(x)}{T_j(x)} \right]^{-1}
\]

• Average over all points

• Average over a range of k
Step 3: Discuss Properties of MLE

• Expected value of MLE = D

• Variance = D^2/(k-3)

• These are asymptotic for k and sample size

Results

Critique

• GOOD
  – Provides a well-defined tool for estimating dimensionality
  – Suitable for dimensions appropriate for visualizing

• BAD
  – Written by Statisticians
  – Absolutely no appeal to intuition
  – No geometric description of Estimator!
Questions?