1, $\begin{pmatrix} 0.35 & 0 & 0 & 0 \\ 0 & 0.35 & 0 & 0 \\ 0 & 0 & 0.35 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Scaling an object by 35%
$\Rightarrow$ (35% X, 35% Y, 35% Z)

2 Can break the matrix into two matrices

$\cos 60° = 0.5 \qquad \sin 60° = 0.866$

$$R = \begin{pmatrix} .5 & -.866 & 0 & 0 \\ .866 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

rotation around +Z axis by 60°

$$T = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

translate by 2 in X direction and -3 in Y direction

which order?

if T then R $\Rightarrow$ RT

$$\begin{pmatrix} .5 & -.866 & 0 & 0 \\ .866 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .5 & -.866 & 0 & 3.598 \\ .866 & .5 & 0 & 0.232 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is NOT equal to the original matrix

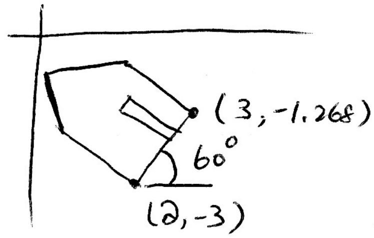
If R then T $\Rightarrow$ TR

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} .5 & -.866 & 0 & 0 \\ .866 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .5 & -.866 & 0 & 2 \\ .866 & .5 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

this is the order we want
we read the matrices from RIGHT to LEFT

$\therefore$ rotation around +Z axis for 60° then translate by 2 in X direction and -3 in Y direction.

3,



We do rotation and translate with respect to the WORLD coordinate. If you are not sure, you can always use a point to try.

ie.
$$\begin{pmatrix} .5 & -.866 & 0 & 2 \\ .866 & .5 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1.268 \\ 0 \\ 1 \end{pmatrix}$$

4, 3 Steps

① Move point of rotation to origin

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ translate by (-3, -5, -12)

② Rotate around +X axis by 30°

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\sin 30 = 0.5$
$\cos 30 = 0.866$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & .866 & -.5 & 0 \\ 0 & .5 & .866 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

③ Move point back the same amount

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ translate by (3, 5, 12)

Matrices should be written from right to left. First operation on the RIGHT. Last operation on the LEFT.

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & .866 & -.5 & 0 \\ 0 & .5 & .866 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can check the correctness of our matrix by using the fact that the centre of rotation doesn't move.

If we multiply our matrix to $(3, 5, 12, 1)^T$, we should get $(3, 5, 12, 1)^T$ back.

5 OpenGL uses post-multiply which mean it multiplies a new matrix to the left of the current one. ie. $M_2 = M_1 M_{new}$
So the commands we give are according to the LEFT to RIGHT ordering.

```
glTranslatef(3, 5, 12)
glRotatef(30, 1, 0, 0)
glTranslatef(-3, -5, -12)
```

Some people may think we don't ~~t~~ need the last command if we draw after the rotation, but that only works if objects are defined in respect to the point (3, 5, 12). Since objects are defined with respect to the origin, we must include the last command. Also, we want these to effect the whole scene, if they're encapsulated in push/pop matrix, then there's no effect at all.

6. It is sufficient to show the 2D case.

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Inverse of R is ~~equal~~ equivalent to rotate by $-\theta$

$$R^{-1}(\theta) = R(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

since $\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

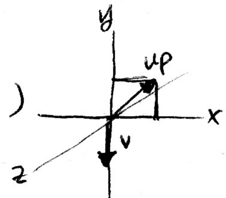
$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= R^T(\theta)$$

7, 3 steps

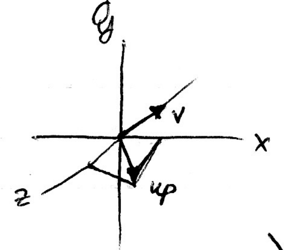
① translate eye to origin

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ translate by $(0,-1,0)$



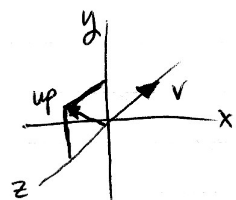
② bring view vector to align with $-Z$ (by rotating 90° around $+X$ axis)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sin 90^\circ = +1 \quad \cos 90^\circ = 0 \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



③ bring up vector into YZ plane (by rotating 90° around $+Z$ axis)

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sin 90^\circ = 1 \quad \cos 90^\circ = 0 \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Combine all 3. Remember, first operation on ~~LEFT~~ RIGHT, last operation on ~~RIGHT~~ LEFT.

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8.
$$P = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$n = 1 \quad f = 2$

if $l = -1 \quad r = 1$

$b = -1 \quad t = 1$

$$P = \begin{pmatrix} \frac{2(1)}{1-(-1)} & 0 & \frac{1+(-1)}{1-(-1)} & 0 \\ 0 & \frac{2(1)}{(1)-(-1)} & \frac{1+(-1)}{1-(-1)} & 0 \\ 0 & 0 & \frac{-(2+1)}{2-1} & \frac{-2(2)(1)}{2-1} \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

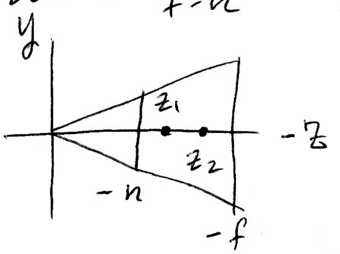
9. Using P above

$$\begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{pmatrix} X' \\ Y' \\ Z' \\ -Z \end{pmatrix} \qquad Z' = -\frac{(f+n)}{f-n} Z - \frac{2fn}{f-n}$$

normalize by dividing by $w = -Z$

$$Z'' = \frac{f+n}{f-n} + \frac{2fn}{f-n}\left(\frac{1}{Z}\right)$$

Let $\alpha = \frac{f+n}{f-n} > 0$, $\beta = \frac{2fn}{f-n} > 0 \qquad Z'' = \alpha + \frac{\beta}{Z}$



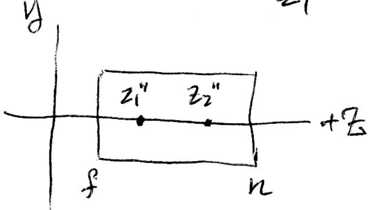
Z_2 is further away from Z_1

since they are negative, $Z_1 > Z_2$

using Z'', $\quad Z_1'' = \alpha + \frac{\beta}{Z_1} \qquad Z_2'' = \alpha + \frac{\beta}{Z_2}$

Since $Z_1 > Z_2 \Rightarrow \frac{1}{Z_1} < \frac{1}{Z_2} \Rightarrow \frac{\beta}{Z_1} < \frac{\beta}{Z_2}$

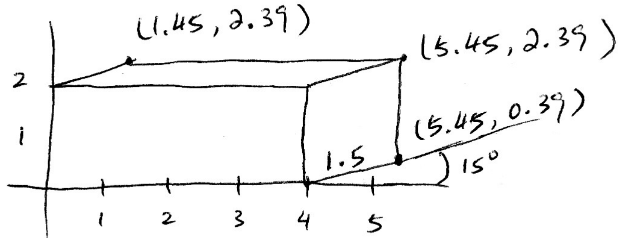
$$\Rightarrow \alpha + \frac{\beta}{Z_1} < \alpha + \frac{\beta}{Z_2} \Rightarrow Z_1'' < Z_2''$$

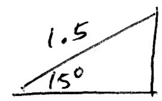


$\therefore Z_2''$ is further away from Z_1''

$\Rightarrow$ the order of Z values are preserved

10,





$y = 1.5 \sin 15^\circ = 0.39$

$X = 1.5 \cos 15^\circ = 1.45$

Cabinet projection's Z is half the actual value.

11, All points are mapped to $Z=0$ plane.

First method that always work

Pick 5 points

$(0,0,0,1)^T \rightarrow (0,0,0,1)^T$

$(4,2,0,1)^T \rightarrow (4,2,0,1)^T$

$(4,2,-3,1)^T \rightarrow (5.45, 2.39, 0, 1)^T$

$(0,2,-3,1)^T \rightarrow (1.45, 2.39, 0, 1)^T$

$(4,0,-3,1)^T \rightarrow (5.45, 0.39, 0, 1)^T$

Using these 5 points to solve for matrix P

$$P = \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & P & 1 \end{pmatrix}$$

we get

$$\begin{pmatrix} D \\ H \\ L \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 4A+2B+D \\ 4E+2F+H \\ 4I+2J+L \\ 4M+2N+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 4A+2B-3C+D \\ 4E+2F-3G+H \\ 4I+2J-3K+L \\ 4M+2N-3P+1 \end{pmatrix} = \begin{pmatrix} 5.45 \\ 2.39 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2B-3C+D \\ 2F-3G+H \\ 2J-3K+L \\ 2N-3P+1 \end{pmatrix} = \begin{pmatrix} 1.45 \\ 2.39 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 4A-3C+D \\ 4E-3G+H \\ 4I-3K+L \\ 4M-3P+1 \end{pmatrix} = \begin{pmatrix} 5.45 \\ 0.39 \\ 0 \\ 1 \end{pmatrix}$$

We can solve for the first row of the matrix using the following equations

① $D = 0$

② $4A + 2B + D = 4$

③ $4A + 2B - 3C + D = 5.45$

④ $2B - 3C + D = 1.45$

⑤ $4A - 3C + D = 5.45$

③ − ② ⇒ $-3C = 1.45$

$C = -0.48$

③ − ④ ⇒ $4A = 4$

$A = 1$

③ − ⑤ ⇒ $2B = 0$

$B = 0$

① $D = 0$

∴ first row $= (1 \quad 0 \quad -0.48 \quad 0)$

Second row

① $H = 0$

② $4E + 2F + H = 2$

③ $4E + 2F - 3G + H = 2.39$

④ $2F - 3G + H = 2.39$

⑤ $4E - 3G + H = 0.39$

③ − ② ⇒ $-3G = 0.39$

$G = -0.13$

③ − ④ ⇒ $4E = 0$

$E = 0$

③ − ⑤ ⇒ $2F = 2$

$F = 1$

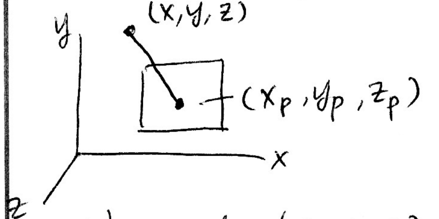
① $H = 0$

∴ Second row $= (0 \quad 1 \quad -0.13 \quad 0)$

Solve for third and fourth row similarly, the final matrix is

$$P = \begin{pmatrix} 1 & 0 & -0.48 & 0 \\ 0 & 1 & -0.13 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Second method from the book

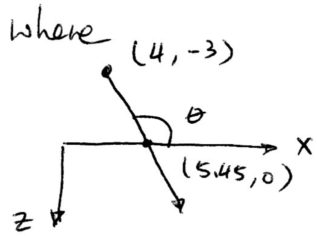


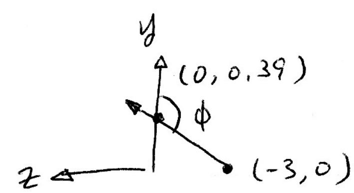
We need $(x, y, z) \Rightarrow (x_p, y_p, z_p)$

use $(4, 0, -3) \Rightarrow (5.45, 0.39, 0)$

$$P = \begin{pmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where





$x_p = x - z\cot\theta$

$5.45 = 4 - (-3)\cot\theta$

$5.45 - 4 = 3\cot\theta$

$\frac{1.45}{3} = \cot\theta$

$\cot\theta = 0.48$

$y_p = y - z\cot\phi$

$0.39 = 0 - (-3)\cot\phi$

$0.39 = 3\cot\phi$

$\frac{0.39}{3} = \cot\phi$

$\cot\phi = 0.13$

$$P = \begin{pmatrix} 1 & 0 & -0.48 & 0 \\ 0 & 1 & -0.13 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

θ and ϕ are not related to 15° directly,
15° is used to determine to point (x_p, y_p, z_p)