University of British Columbia

CPSC 414 Computer Graphics

Rotations and Quaternions

Week 9, Wed 29 Oct 2003

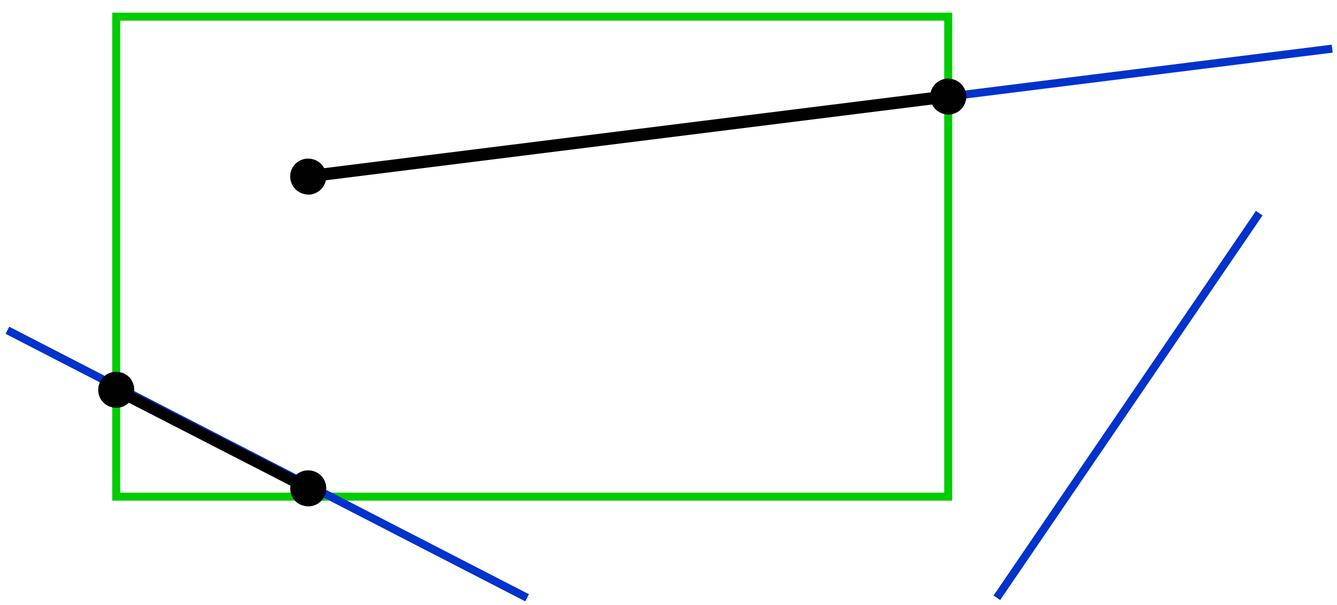
University of British Columbia

CPSC 414 Computer Graphics

Clipping recap

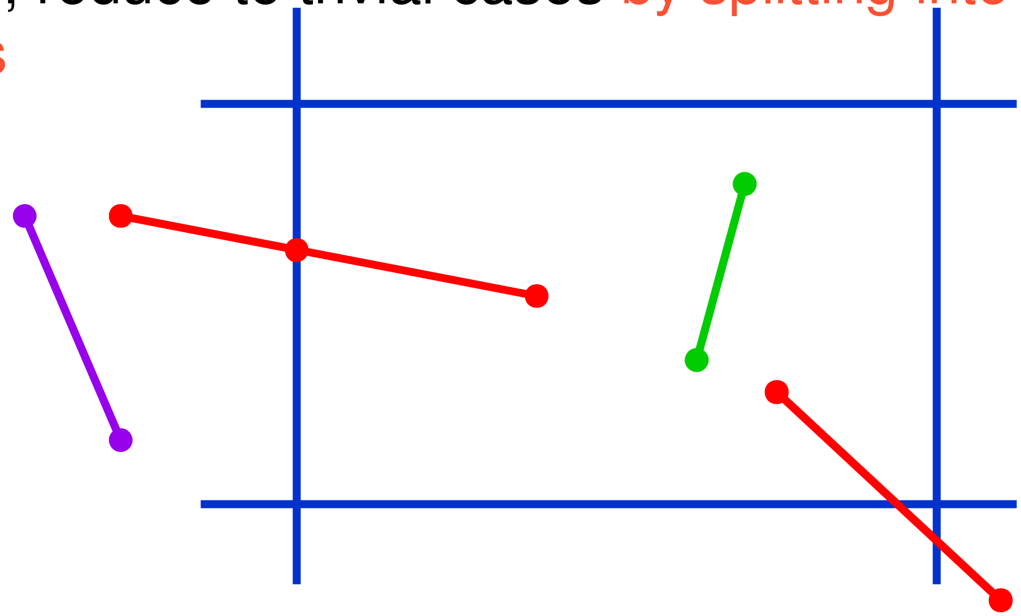
Clipping

- analytically calculating the portions of primitives within the viewport



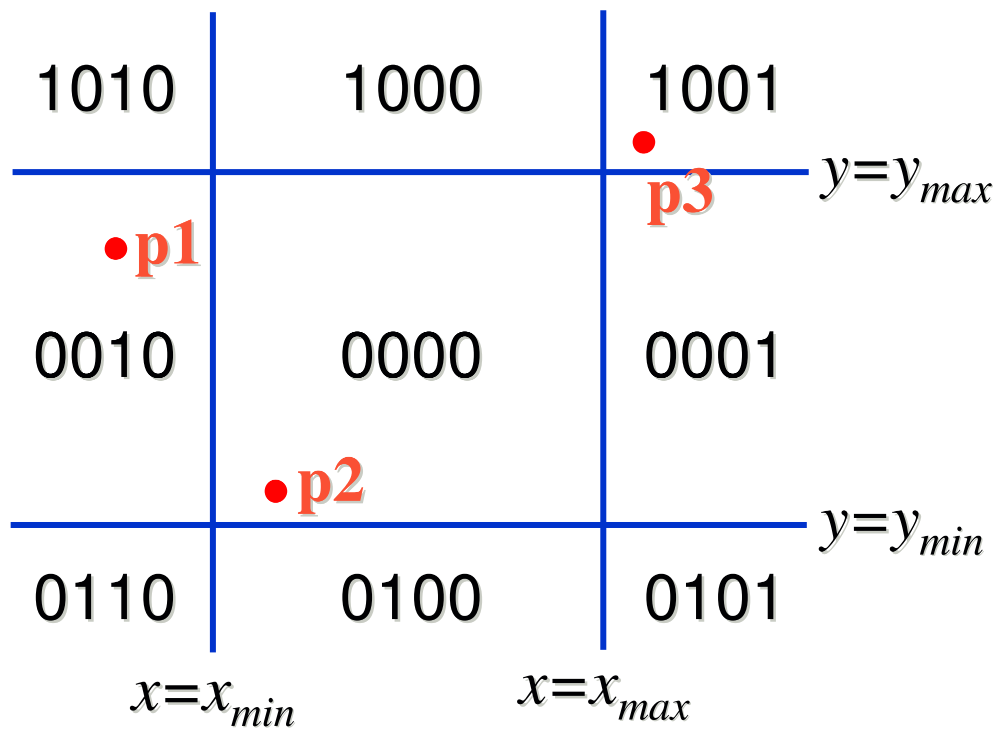
Clipping Lines To Viewport

- combining trivial accepts/rejects
  - trivially accept lines with both endpoints inside all edges of the viewport
  - trivially reject lines with both endpoints outside the same edge of the viewport
  - otherwise, reduce to trivial cases by splitting into two segments



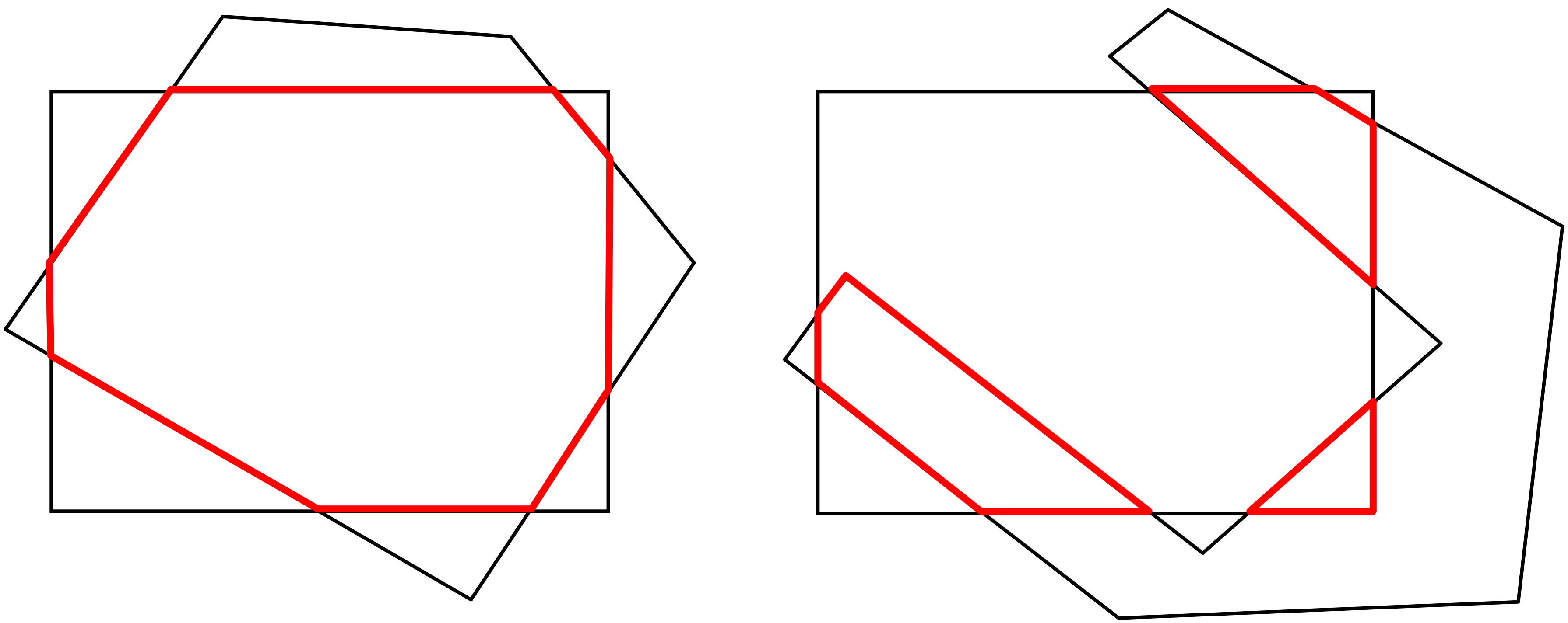
Cohen-Sutherland Line Clipping

- outcodes
  - 4 flags encoding position of a point relative to top, bottom, left, and right boundary
    - *OC(**p1**)=0010*
    - *OC(**p2**)=0000*
    - *OC(**p3**)=1001*



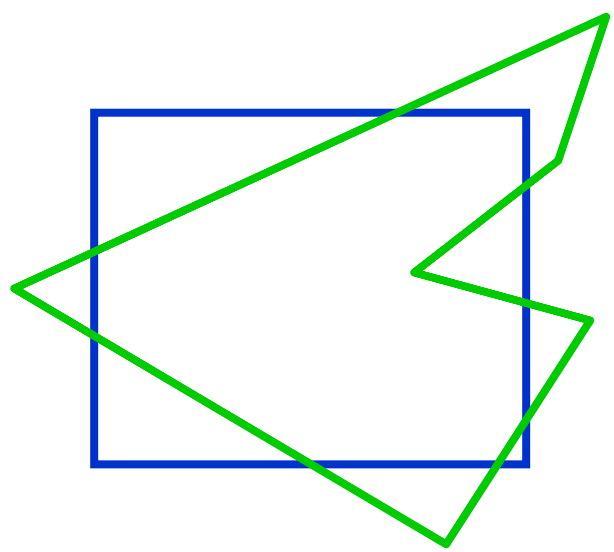
Polygon Clipping

- not just clipping all boundary lines
  – may have to introduce new line segments



Sutherland-Hodgeman Clipping

- basic idea:
  - consider each edge of the viewport individually
  - clip the polygon against the edge equation
  - after doing all edges, the polygon is fully clipped



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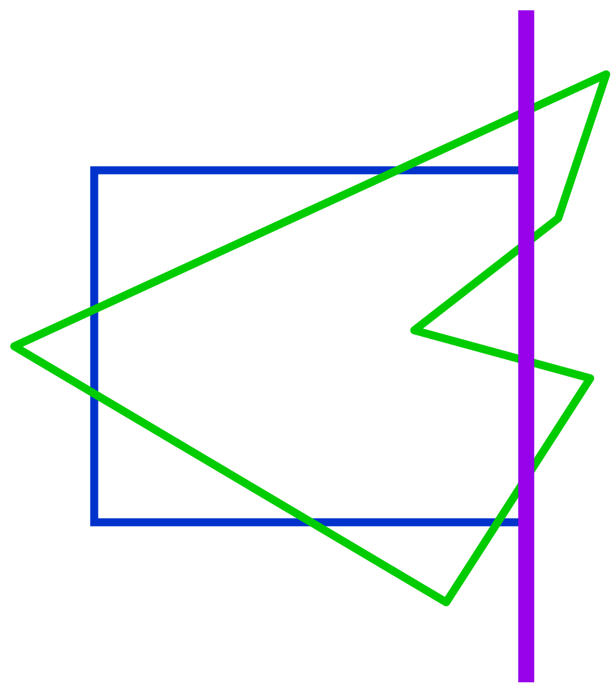

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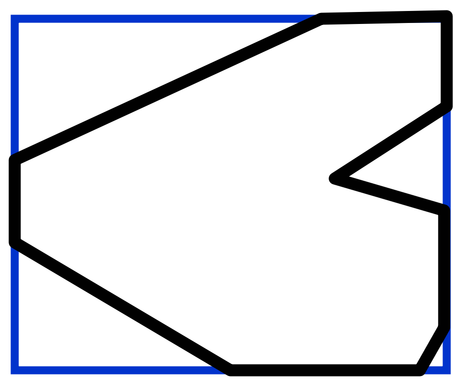
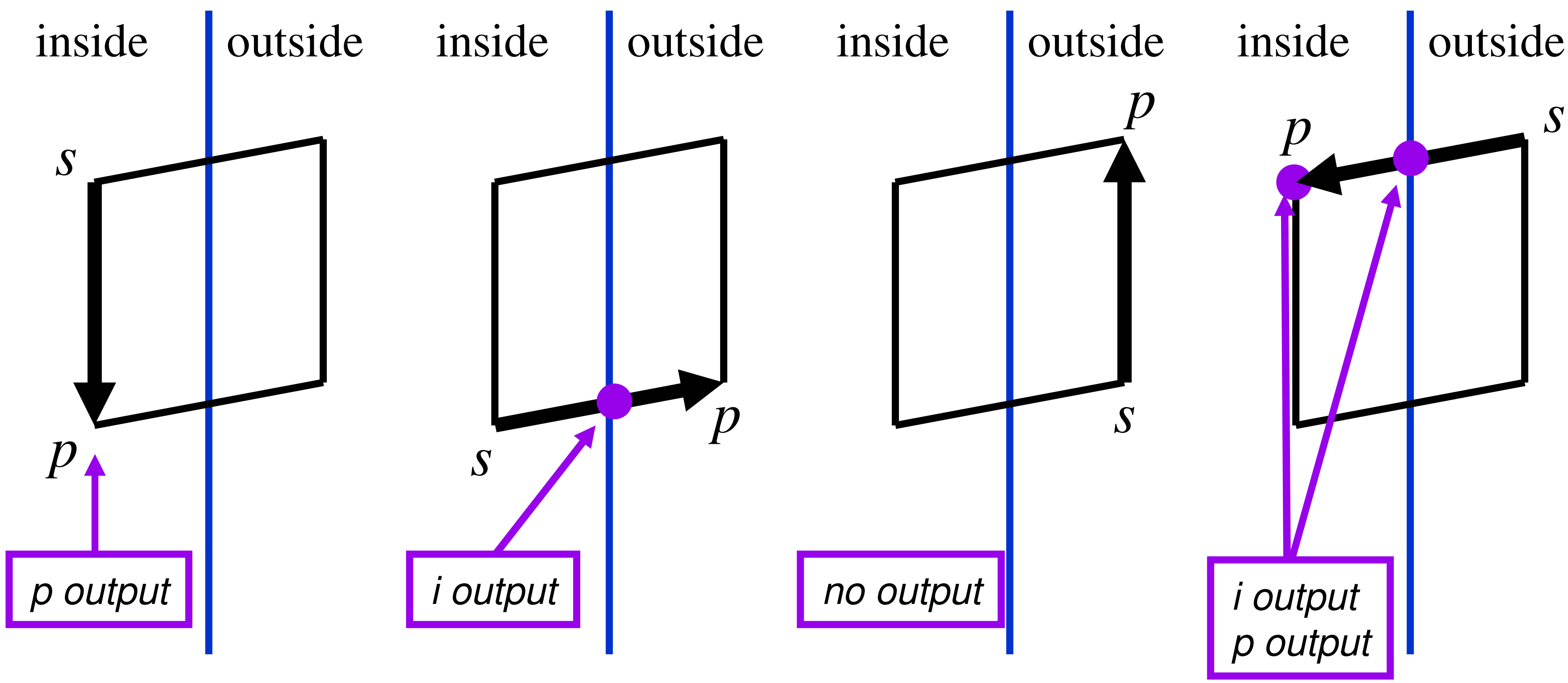


Sutherland-Hodgeman Clipping

- edge from *s* to *p* takes one of four cases:

  (blue line can be a line or a plane)



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CPSC 414 Computer Graphics

Rotations and Quaternions

Camera Movement Hints

- change viewing transformation
  - don’t try doing this with perspective xform!
- methods
  - gluLookAt
  - direct camera control using rotate/translate
- camera motion opposite of object motion
  - rotate world by a = orbit camera by -a

Parameterizing Rotations

- Straightforward in 2D
  - A scalar, θ, represents rotation in plane
- More complicated in 3D
  - Three scalars are required to define orientation
  - Note that three scalars are also required to define position
  - Objects free to translate and tumble in 3D have 6 degrees of freedom (DOF)

Representing 3 Rotational DOFs

- 3x3 Matrix (9 DOFs)
  - Rows of matrix define orthogonal axes
- Euler Angles (3 DOFs)
  - Rot x + Rot y + Rot z
- Axis-angle (4 DOFs)
  - Axis of rotation + Rotation amount
- Quaternion (4 DOFs)
  - 4 dimensional complex numbers

3x3 Rotation Matrix

- 9 DOFs must reduce to 3
- Rows must be unit length (-3 DOFs)
- Rows must be orthogonal (-3 DOFs)
- Drifting matrices is very bad
  - Numerical errors results when trying to gradually rotate matrix by adding derivatives
  - Resulting matrix may scale / shear
  - Gram-Schmidt algorithm will re-orthogonalize
- Difficult to interpolate between matrices
  - How would you do it?

Rotation Matrix

- general rotation can be represented by a single 3x3 matrix
  - length preserving (isometric)
  - reflection preserving
  - orthonormal

$$R = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

- problem:
  - property: rows and columns are orthonormal (unit length and perpendicular to each other)
  - linear interpolation doesn't maintain this property

Rotation Matrices Not Interpolatable

- interpolate linearly from +90 to -90 in y

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- halfway through component interpolation

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  – problem 1: not a rotation matrix anymore!
    - not orthonormal, x flattened out

Euler Angles

- $(\theta_x, \theta_y, \theta_z) = R_z R_y R_x$
  - Rotate θ_x degrees about x-axis
  - Rotate θ_y degrees about y-axis
  - Rotate θ_z degrees about z-axis
- Axis order is not defined
  - (y, z, x), (x, z, y), (z, y,x)… all legal
  - Pick one

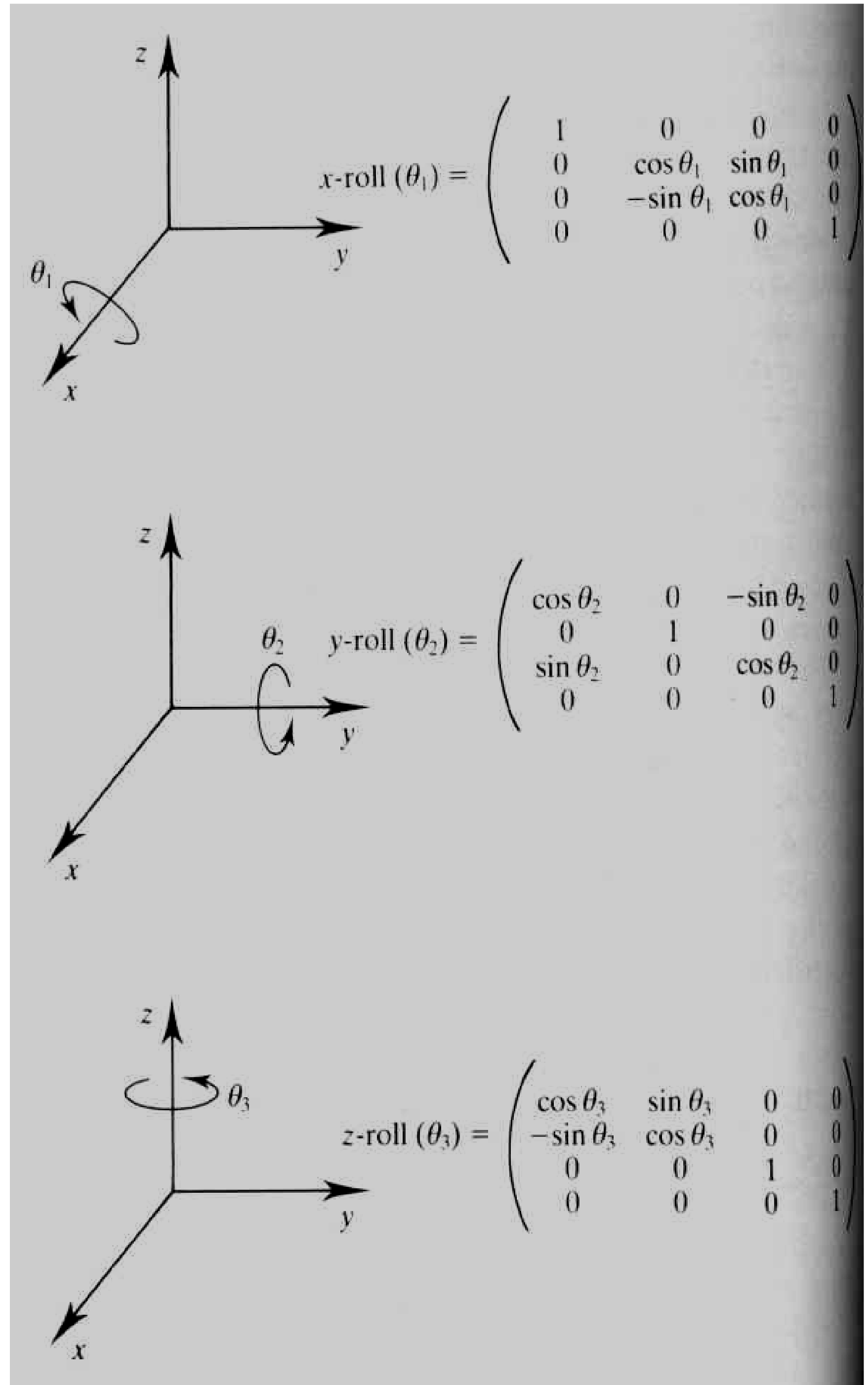


Figure 15.15 The principal rotation matrices.

Euler Angle Interpolation

- solution 1: can interpolate angles individually
- problem 2: interpolation between two Euler angles is not unique
- ex: (x, y, z) rotation
  - (0, 0, 0) to (180, 0, 0) vs. (0, 0, 0) to (0, 180, 180)
  - interpolation about different axes are not independent
  - Cartesian coordinates are independent of one another, but Euler angles are not

Interpolation

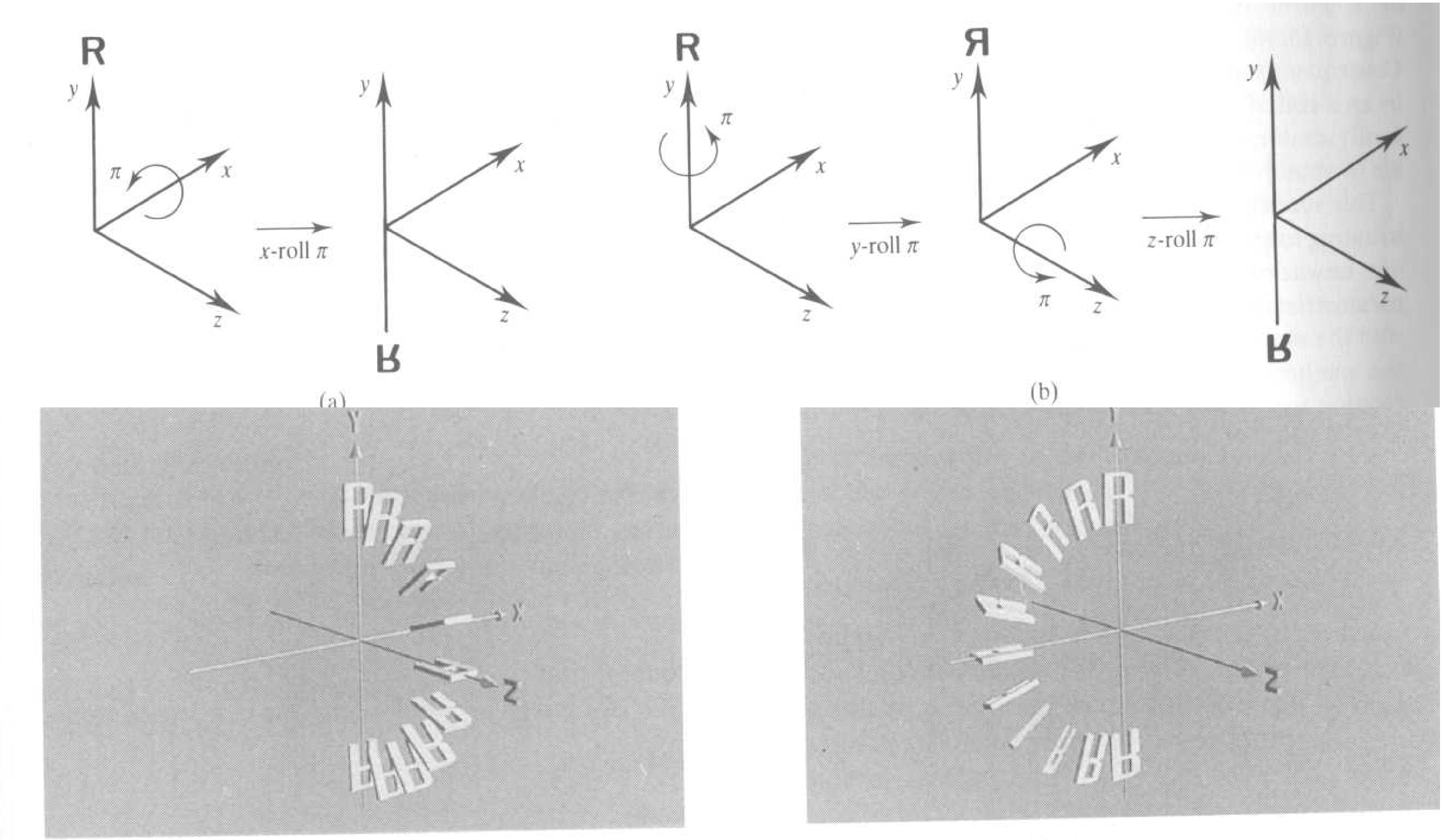
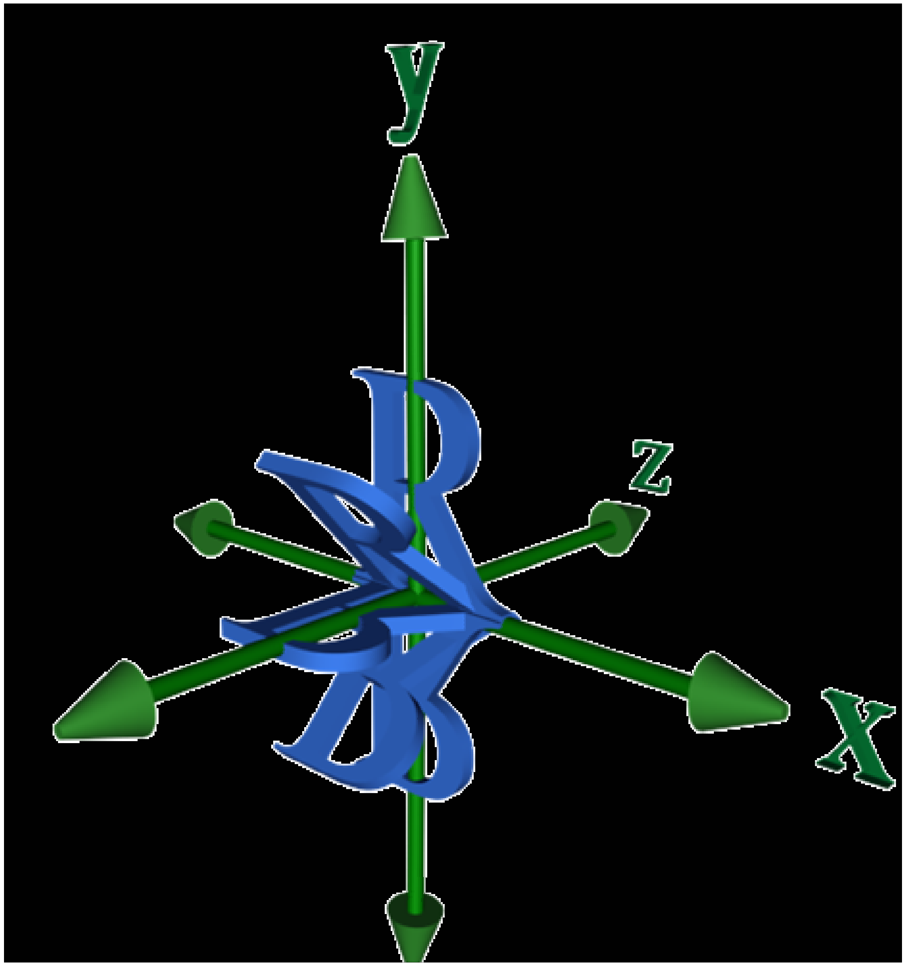
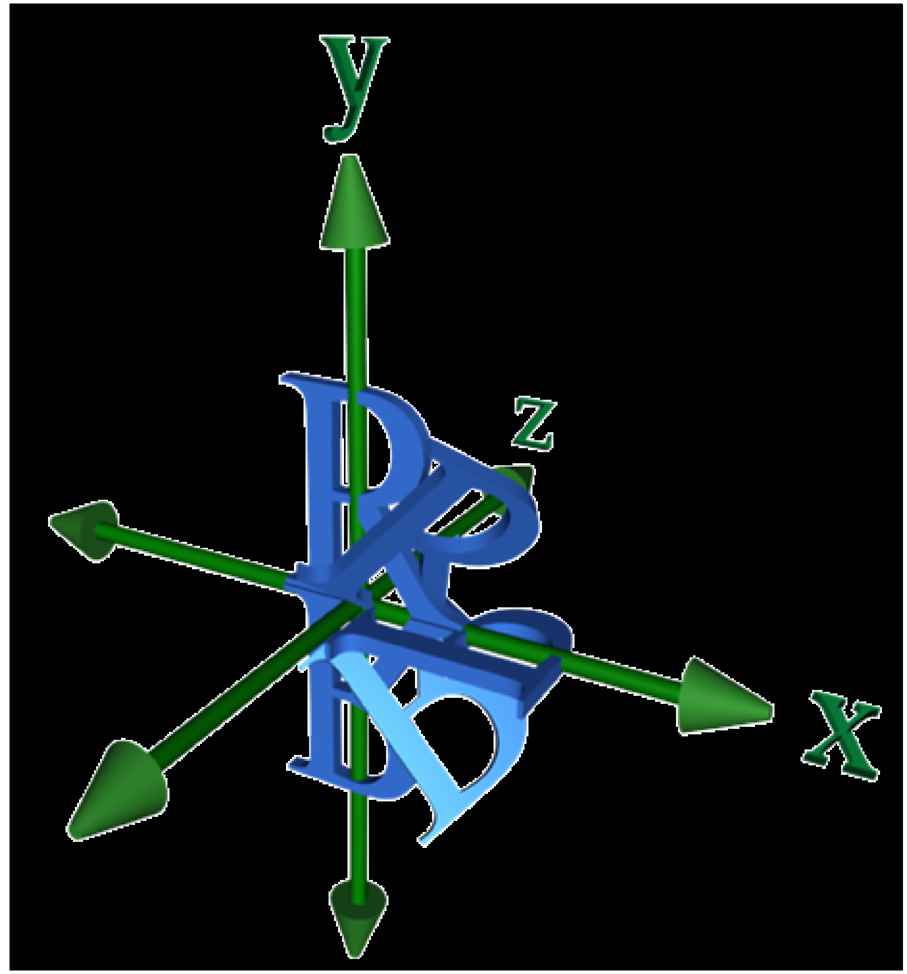


Figure 15.19 Euler angle parametrization. (a) A single x-roll of π. (b) A y-roll of π followed by a z-roll of π.

Interpolation

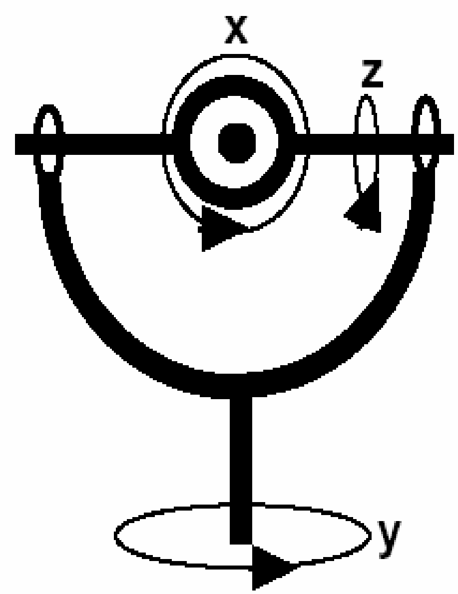




Euler Angles

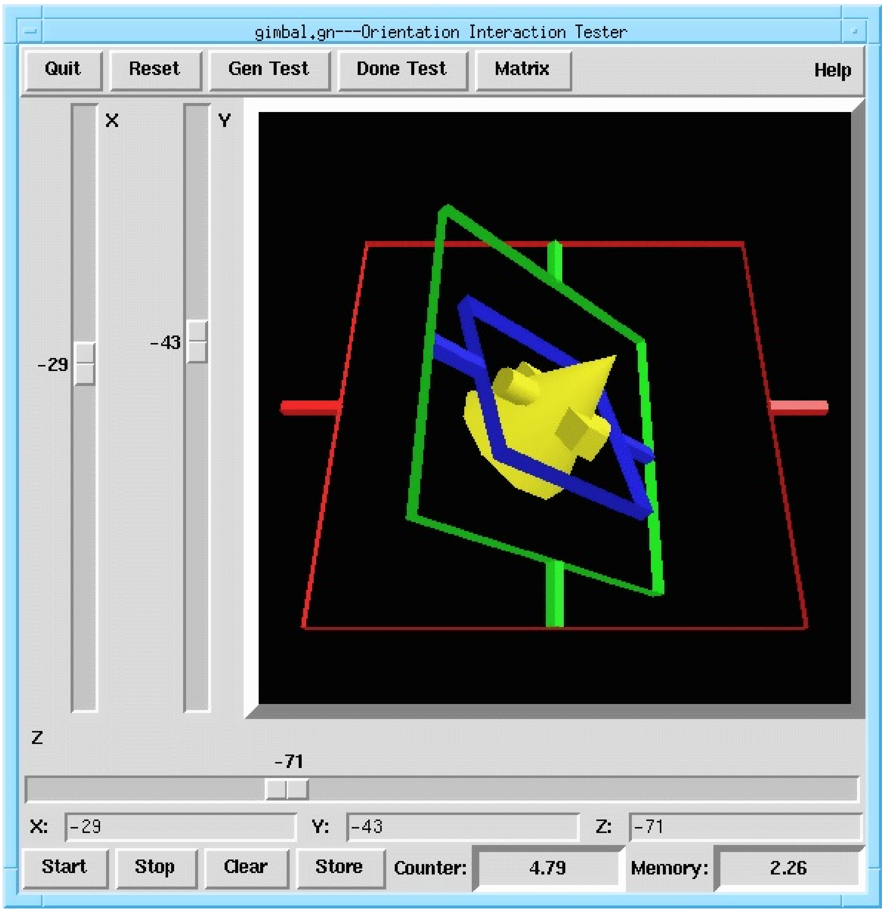
- Problem 3: Gimbal Lock
  - term derived from mechanical problem that arises in gimbal mechanism that supports a compass or a gyro

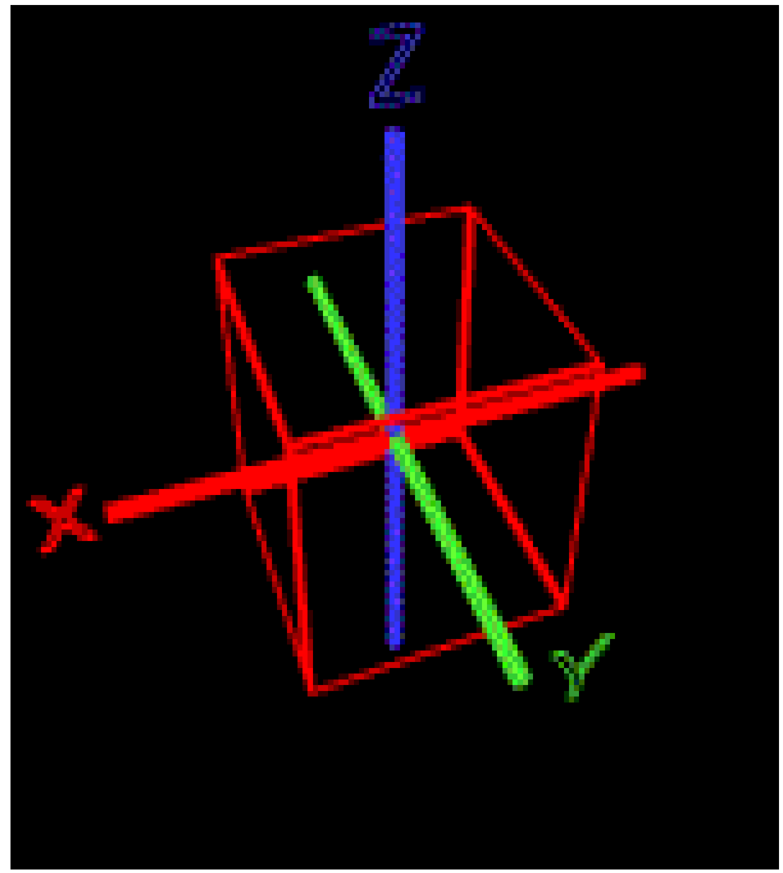
gimbal: hardware implementation of Euler angles (used for mounting gyroscopes and globes)



Gimbal

Gimbal Lock



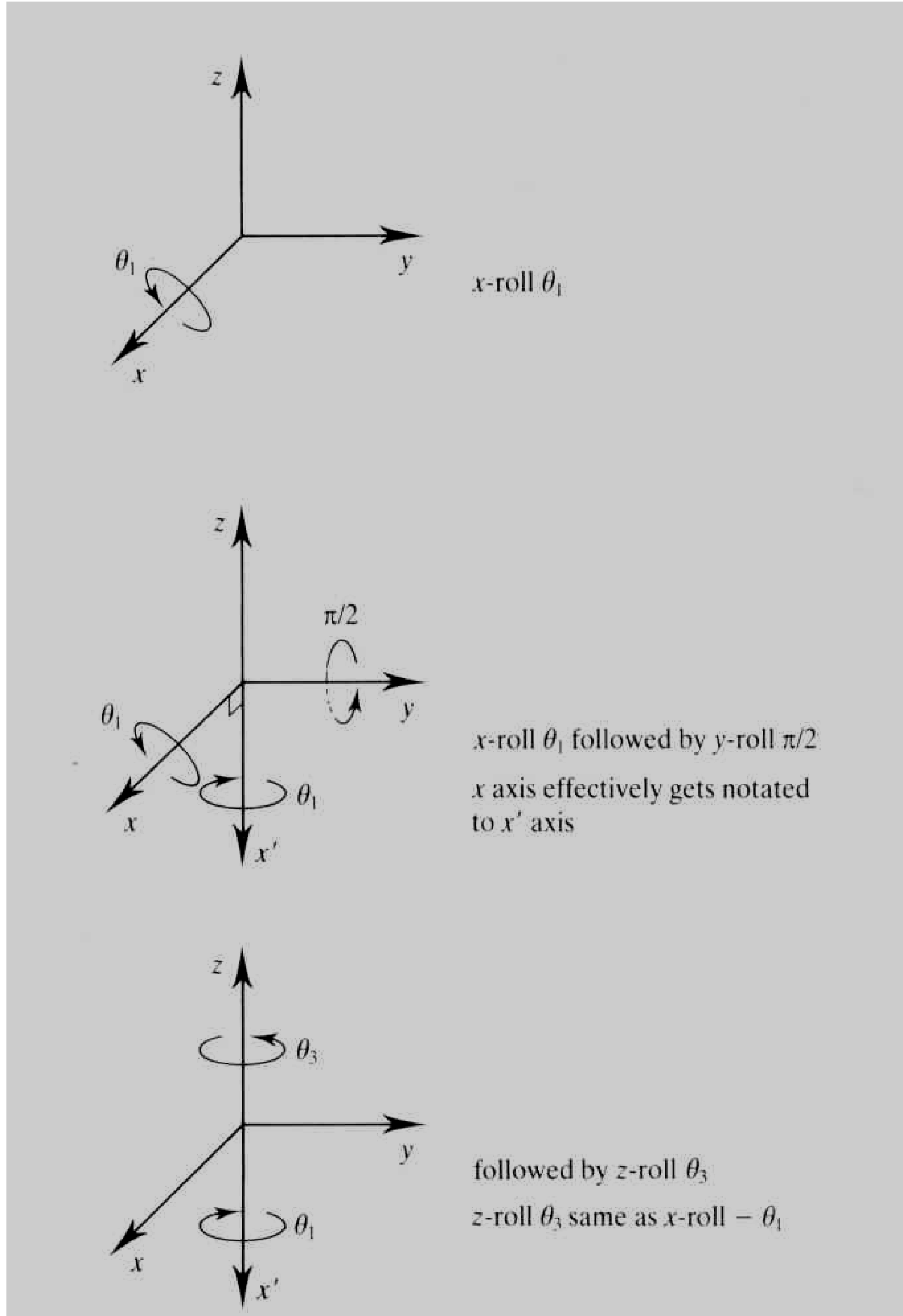


http://www.anticz.com/eularqua.htm

Gimbal Lock

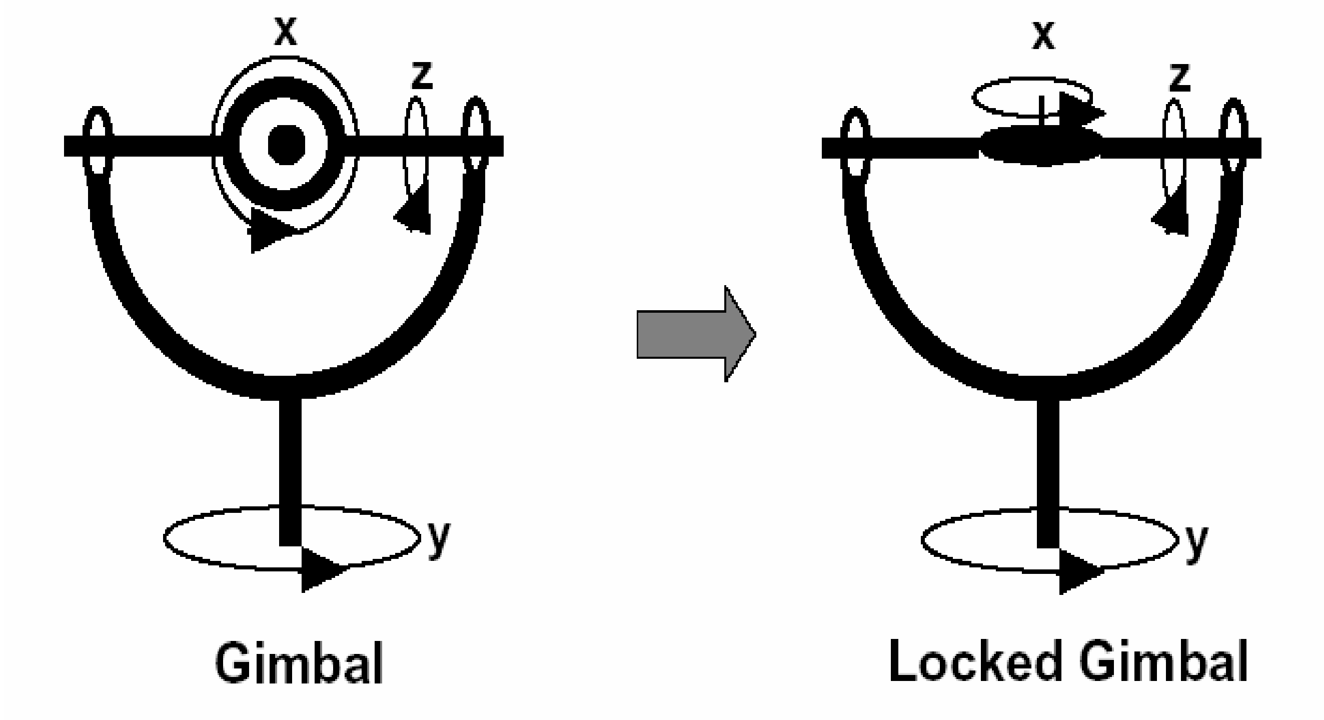
- Occurs when two axes are aligned
- Second and third rotations have effect of transforming earlier rotations
  - If Rot y = 90 degrees, Rot z == -Rot x

[demo]

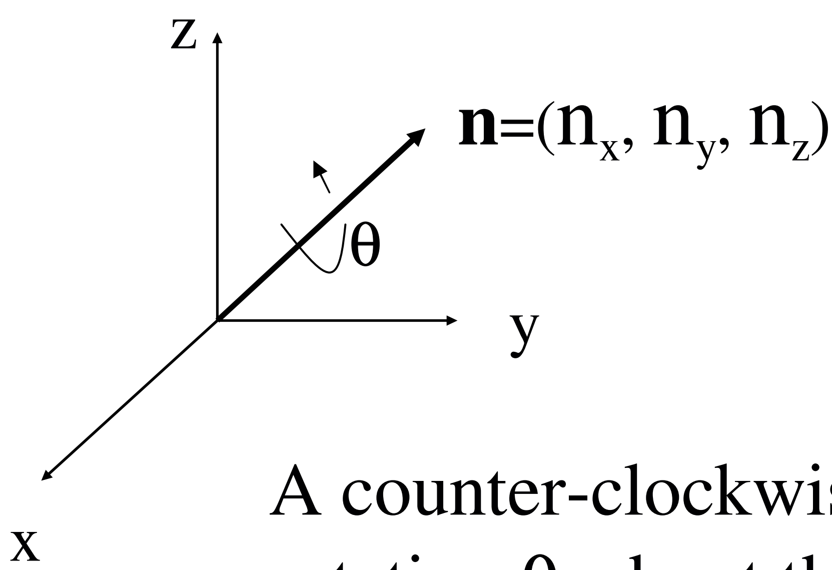


Locked Gimbal

- ***Hardware implementation of Euler angles (used for mounting gyroscopes and globes)***



Axis-angle Rotation



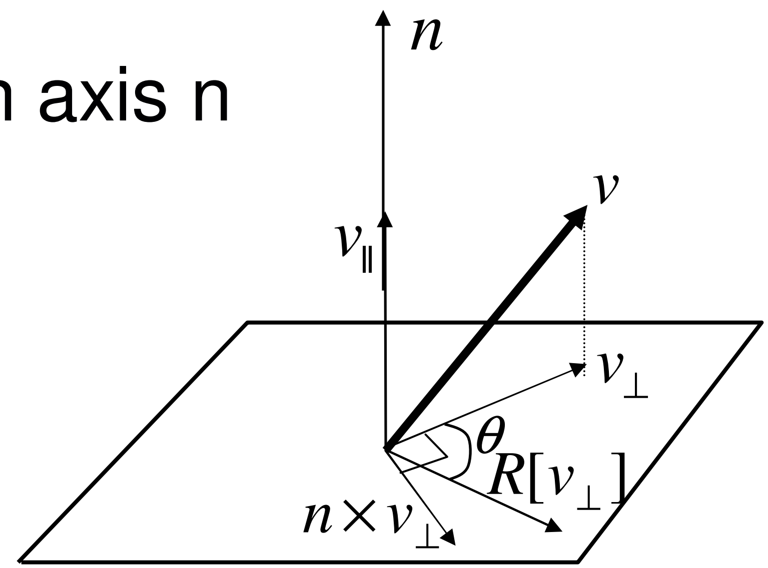
A counter-clockwise (right-handed) rotation θ about the axis specified by the unit vector $\mathbf{n}=(n_x, n_y, n_z)$

Axis-angle Notation

- Define an axis of rotation (x, y, z) and a rotation about that axis, θ: R(θ, **n**)
- 4 degrees of freedom specify 3 rotational degrees of freedom because axis of rotation is constrained to be a unit vector

Angular displacement

- (θ,n) defines an angular displacement of θ about an axis n



$$v_{\parallel} = (n\cdot v)n \qquad v_{\perp} = v - v_{\parallel}$$

$$R[v_{\perp}] = v_{\perp}\cos\theta + (n\times v_{\perp})\sin\theta$$

$$= v_{\perp}\cos\theta + (n\times v)\sin\theta$$

$$R[v_{\parallel}] = v_{\parallel}$$

$$R[v] = R[v_{\parallel} + v_{\perp}] = R[v_{\parallel}] + R[v_{\perp}] = v_{\parallel} + v_{\perp}\cos\theta + (n\times v)\sin\theta$$

$$= (n\cdot v)n + (v - (n\cdot v)n)\cos\theta + (n\times v)\sin\theta$$

$$= v\cos\theta + n(n\cdot v)(1-\cos\theta) + (n\times v)\sin\theta$$

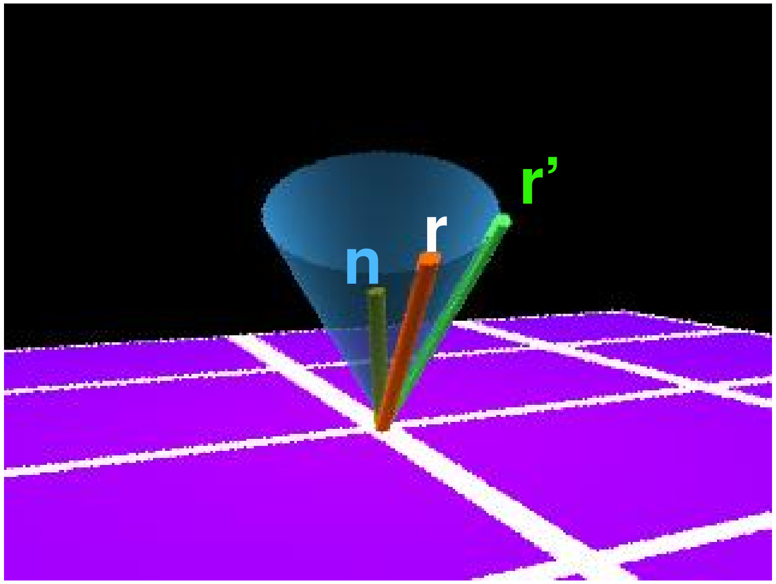
Axis-angle Rotation

Given

- r – Vector in space to rotate
- n – Unit-length axis in space about which to rotate
- θ – The amount about n to rotate

Solve

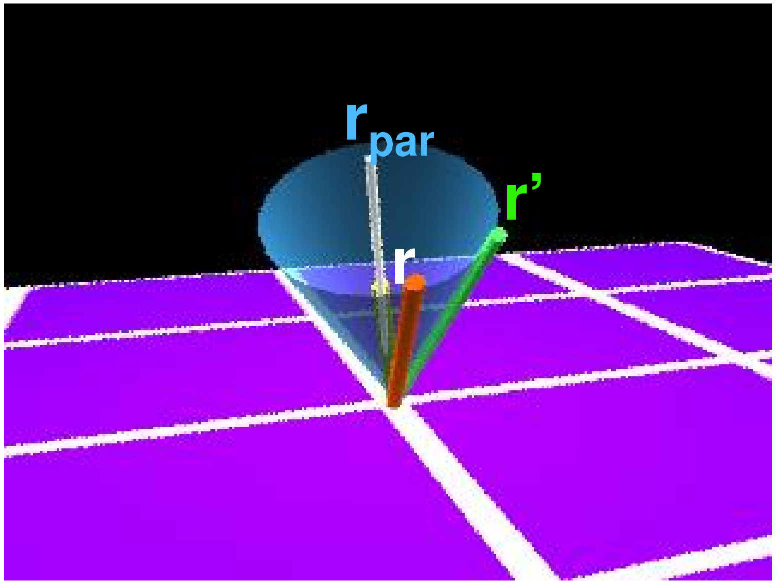
- r' – The rotated vector



Axis-angle Rotation

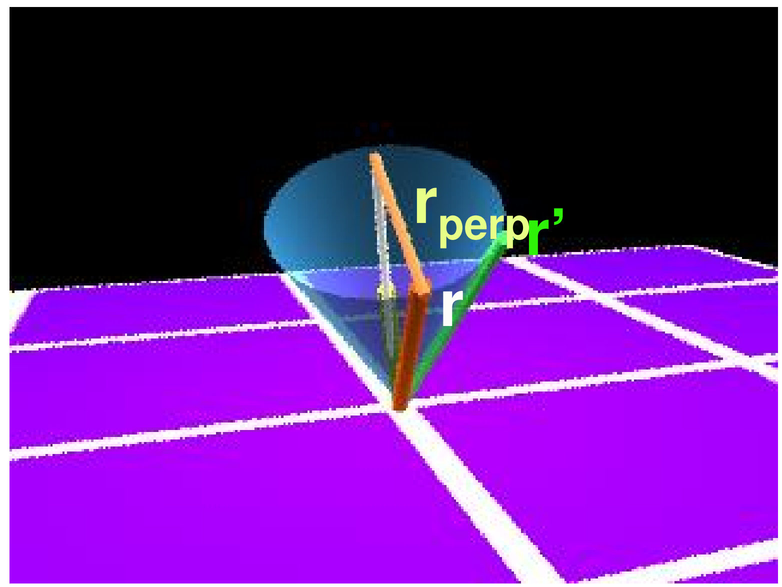
- step 1
  - compute r_{par} , an extended version of the rotation axis n

r_{par} = (n . r) n



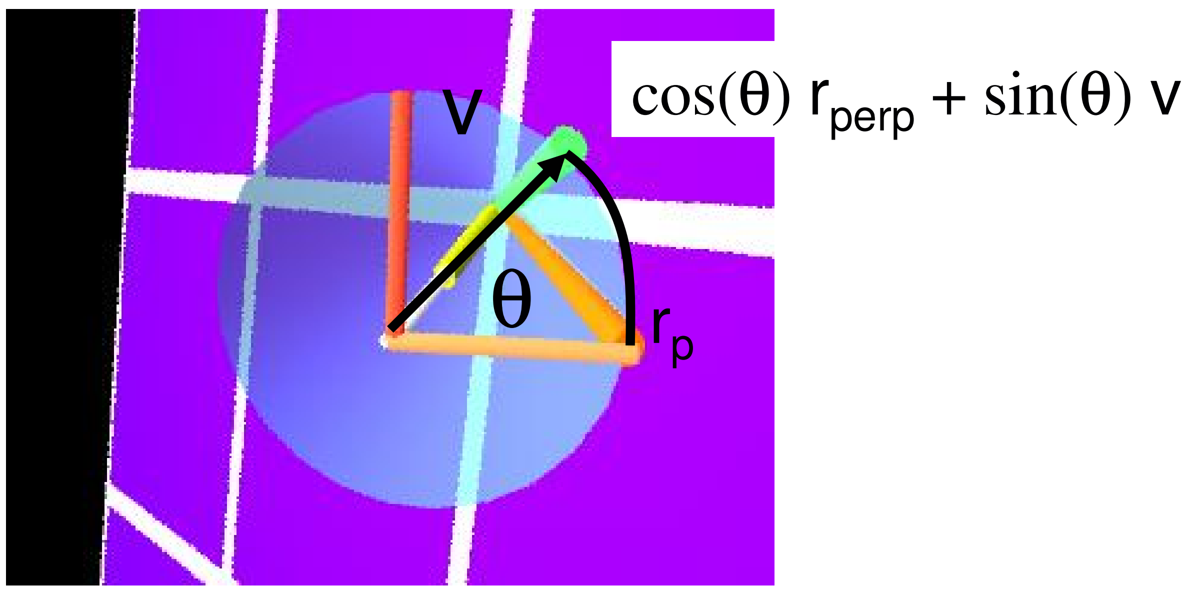
Axis-angle Rotation

- Compute r_{perp}
- $r_{perp} = r - r_{par} =$
  $r - (n \,.\, r)\, n$



Axis-angle Rotation

- Compute v, a vector perpendicular to r_{par} , r_{perp}
- Use v and r_{perp} and θ to compute r'



Angular Displacement

- $R(\theta, \mathbf{n})$ is the rotation matrix to apply to a vector $\mathbf{v}$, then,

  $$R[v]=v\cos\theta + n(n.v)(1-\cos\theta) + (n\times v)\sin\theta$$

  – It guarantees a simple steady rotation between any two key orientations

  – It defines moves that are independent of the choice of the coordinate system

Axis-angle Notation

- solutions
  - any orientation can be represented by a 4-tuple angle, vector(x,y,z)
  - can interpolate the angle and axis separately
  - no gimbal lock problems!
- problems
  - no easy way to determine how to concatenate many axis-angle rotations that result in final desired axis-angle rotation
    - so can't efficiently compose rotation, must convert to matrices first!

Quaternions

- extend the concept of rotation in 3D to 4D
- avoids the problem of "gimbal-lock" and allows for the implementation of smooth and continuous rotation
- in effect, they may be considered to add a additional rotation angle to spherical coordinates ie. longitude, latitude and rotation angles
- a quaternion is defined using four floating point values |x y z w|. These are calculated from the combination of the three coordinates of the rotation axis and the rotation angle.

Quaternions Definition

- Extension of complex numbers
- 4-tuple of real numbers
  - s,x,y,z or [s,v]
  - s is a scalar
  - v is a vector
- Same information as axis/angle but in a different form
- Can be viewed as an original orientation or a rotation to apply to an object

Quaternion

- Extension of complex numbers: $a + ib$
  – remember $i^2 = -1$
- Quaternion:
  – $Q = a + bi + cj + dk$
    - Where $i^2 = j^2 = k^2 = -1$ and $ij = k$ and $ji = -k$
  – Represented as: $q = (s, \mathbf{v}) = s + v_x i + v_y j + v_z k$
- Invented by Sir William Hamilton (1843)
  – carved equation into Dublin bridge when discovered after decade of work

Quaternion

- A quaternion is a 4-D unit vector q = [x y z w]
  - It lies on the unit hypersphere $x^2 + y^2 + z^2 + w^2 = 1$
- For rotation about (unit) axis v by angle θ
  - vector part = $(\sin \theta/2)\ v$ = *[x y z]*
  - scalar part = $(\cos \theta/2)$ = *w*
  - $(\sin(\theta/2)\ n_x, \sin(\theta/2)\ n_y, \sin(\theta/2)\ n_{z,} \cos(\theta/2))$
- Only a unit quaternion encodes a rotation
  - must normalize!

Quaternion

- Rotation matrix corresponding to a quaternion:
  - [x y z w] =

$$\begin{bmatrix} 1-2y^2-2z^2 & 2xy+2wz & 2xz-2wy \\ 2xy-2wz & 1-2x^2-2z^2 & 2yz+2wx \\ 2xz+2wy & 2yz-2wx & 1-2x^2-2y^2 \end{bmatrix}$$

- Quaternion Multiplication
  - $q_1 * q_2 = [\mathbf{v_1}, w_1] * [\mathbf{v_2}, w_2] =$
    $[(w_1 v_2 + w_2 v_1 + (v_1 \times v_2)), w_1 w_2 - v_1 . v_2]$
  - quaternion * quaternion = quaternion
  - this satisfies requirements for mathematical *group*
  - Rotating object twice according to two different quaternions is equivalent to one rotation according to product of two quaternions

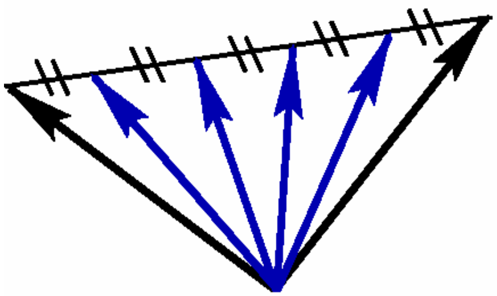
Quaternion Example

- ($\sin(\theta/2)\, n_x$, $\sin(\theta/2)\, n_y$, $\sin(\theta/2)\, n_z$, cos (θ/2))

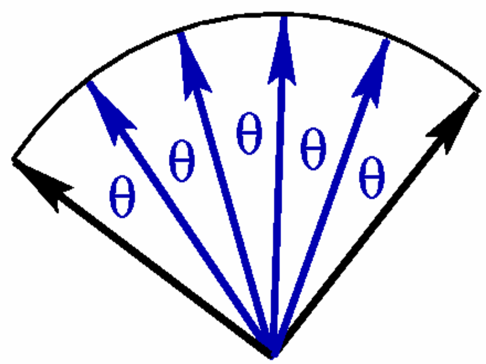
- X-roll of π radians [90 º]
  - (sin (π/2) (1, 0, 0), cos (π/2)) = ((1, 0, 0), 0)
- Y-roll 0f π
  - ((0, 1, 0), 0)
- Z-roll of π
  - ((0, 0, 1), 0)
- R_y (π) followed by R_z (π)
  - ((0, 1, 0), 0) times ((0, 0, 1), 0) =

Quaternion Interpolation

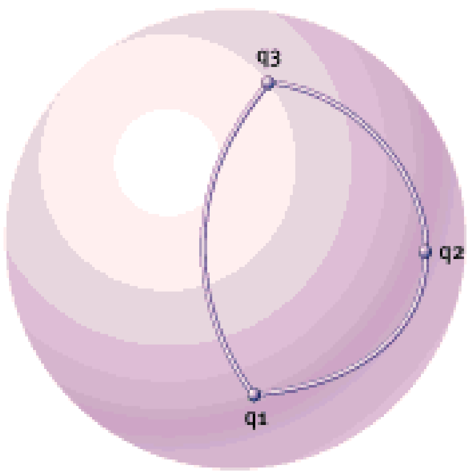
- biggest advantage of quaternions
  - cannot linearly interpolate (lerp) between two quaternions because it would speed up in middle
  - instead, spherical linear interpolation, (slerp)
    - ensure vectors remain on the hypersphere
    - step through using constant angles



Lerping



Slerping



SLERP

- Quaternion is a point on the 4-D unit sphere
  - interpolating rotations requires a unit quaternion at each step
    - another point on the 4-D unit sphere
  - move with constant angular velocity along the great circle between two points
- Any rotation is defined by 2 quaternions, so pick the shortest SLERP
- To interpolate more than two points, solve a non-linear variational constrained optimization
  - Ken Shoemake in SIGGRAPH ’85 (www.acm.org/dl)

Quaternion Libraries

- Gamasutra
  - Code, explanatory article
  - Registration required

http://www.gamasutra.com/features/19980703/quaternions_01.htm

Evaluating Quaternions

- Advantages:
  - Flexible.
  - No parametrization singularities (gimbal lock)
  - Smooth consistent interpolation of orientations.
  - Simple and efficient composition of rotations.
- Disadvantages:
  - Each orientation is represented by two quaternions.
  - Complex!

Summary

- 3x3 matrices
  – drifting, can’t interpolate
- Euler angles
  – gimbal lock
- axis-angle
  – can’t concatenate or interpolate
- quaternions
  – solve all problems, but complex

Project Strategy Suggestion

- debug basics with simple euler angles
  - with single drag, does view change the right way?
- then can add quaternions