Hexahedral Meshing of Non-Linear Volumes Using Voronoi Faces and Edges

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Abstract

This work extends an algorithm presented in our recent paper [1] for automatic hexahedral meshing, based on the embedded Voronoi graph (EVG). The embedded Voronoi graph contains the full symbolic information of the Voronoi diagram and the medial axis of the object, and a geometric approximation to the real geometry. The EVG is used for decomposing the object, into simple sub-volumes meshable by basic meshing techniques.

The EVG provides complete information regarding proximity and adjacency relationships between the entities of the volume. Hence, decomposition faces are determined unambiguously, without any further geometric computations and the resulting sub-volumes are guaranteed to be well-defined and disjoint. The decomposition algorithm is applicable to any volume, including volumes with degenerate medial axis.

The previous paper defined the decomposition based on sub-volumes sweepable perpendicular to an EVG face. This work extends the decomposition to handle all types of EVG entities, providing a complete decomposition of the volume. It analyses the types of sub-volumes that are generated and the meshing techniques applicable to them.

keywords: hexahedra; mesh generation; medial axis; Voronoi diagram; embedded Voronoi graph; decomposition

1 Introduction

Computer simulation of physical properties and behavior of models is essential to reduce the development time for new products. Most of the simulation and analysis techniques used require first to convert the model into a finite element mesh. Various algorithms for mesh generation have been developed (see [2] for a recent review). Several highly successful approaches were developed for tetrahedral meshing [3, 4, 5, 6].

For many physical problems hexahedral meshing provides better solutions than the tetrahedral one. However, the generation of hexahedral meshes turns out to be much more complex than for tetrahedral meshes. There are currently several distinct strategies proposed for unstructured all-hex mesh generation that are predominant in the literature: grid-based [7, 8, 9], whisker weaving [10], medial axis subdivision [11, 12], plastering [13], feature-based [14]. Among the techniques above, only grid-based meshing was shown to be robust, but it tends to generate poor quality elements at the volume boundary and does not preserve mesh conformity along previously meshed faces.

A common approach for semi-automatic meshing of complex geometries is to decompose the volume into sub-part which can then be meshed by algorithms developed for specific types of geometries. These include mappable and sub-mappable volumes [15], volumes meshable by midpoint subdivision [16], swept volumes and uni-axial combinations of swept volumes [17]. The relative success of this approach indicates that a promising direction for solving the general problem is to decompose the volume into parts meshable by existing, well established techniques. This is the approach taken in [14, 11, 12, 18, 1].

In [14], a feature based decomposition of the object is generated. Features are recognized based on combinations of convex and concave edge loops. An attractive property of this approach is that it follows the intuition of manual subdivision. However, the use of feature recognition raises several complicated issues. Features with interacting geometry pose a difficult recognition problem. In many cases there are ambiguities when it is not clear which decomposition the algorithm should choose, and decomposition surfaces can cut each other. The result depends upon the order of performing the decompositions. The method requires computations of surface-surface intersections. Finally, convex shapes are not decomposed at all, even when there is no suitable meshing algorithm for them.

In [11, 12], the decomposition is guided by the medial axis of the object. The proposed algorithm gives a template based decomposition, building a template subdivision around each entity of the medial axis (its faces, edges and vertices). This results in an initial hexahedral mesh of the model, which can then be refined to the desired density.

The use of the medial axis for the decomposition provides a systematic and generic approach for all possible geometries. The decomposition process is directed by the medial axis thus avoiding the computation of decomposing surfaces and surface intersections. However, the described technique has several drawbacks. First, the algorithm does not provide a uniform way for handling medial entities governed by edges and vertices. Second, for degenerate medial edges and vertices the number of governor combinations is unlimited and hence the template approach is not applicable there. Midpoint subdivision can be used for meshing the regions around such entities [12], but this is likely to lead to poorly shaped elements. Third, because the technique builds a template subdivision around each entity of the medial axis, it results in a fine subdivision of the model, even for simple cases. For example, for a brick (with no medial axis degeneracies) the initial mesh contains 72 elements. And last, the algorithm used for computing the medial axis [19] is difficult to implement and not provably correct.

In [1] we suggested decomposing the volume based on the embedded Voronoi graph (EVG) [20]. The embedded Voronoi graph contains the full symbolic information of the Voronoi diagram and the medial axis of the object, and a geometric approximation to the real geometry. The EVG is used for decomposing the object, into simple sub-volumes meshable by basic meshing techniques. Sub-volumes are meshed independently, and the resulting meshes are easily combined and smoothed to yield the final mesh.

This approach possesses several advantages: (1) the strategy is well defined and valid on shapes of any geometry, including shapes whose medial axis is degenerate; (2) the decomposition is order independent and prevents intersections between decomposition surfaces; (3) since the directions and entities involved in each decomposition are defined by the medial axis, there are no intersection computations; (4) it can use the EVG which is an approximation of the medial axis, and not the exact medial axis, and while there is no provably correct algorithm for computing the medial axis, the algorithm for computing the EVG is provably correct, stable and easy to implement [20]; (5) the number of sub-volumes generated is not large since every sub-volume contains a different Voronoi face; (6) since a decomposition is used, as opposed to template, there is only a minimal need for medial axis geometry; and (7) mesh quality seems high since the decomposition avoids generation of sharp angles, and sweep and other basic methods are used to mesh the sub-volumes.

It is worth noticing that while the algorithm presented here can use the exact medial axis instead of the EVG for all the necessary computations, the template based algorithm presented in [12] requires the exact medial axis data and can't use the approximation provided by the EVG.

In [1] the framework for the decomposition procedure is suggested, but the algorithm as presented is not complete. Several areas requiring future research were pointed out. The paper defined the decomposition based on sub-volumes sweepable perpendicular to an EVG face, and did not fully address the meshing of the other parts of the volume. A necessary extension to the presented algorithm is meshing of non-polyhedral volumes. Though a general explanation is given on handling such volumes, it was not fully defined or implemented.

Contribution.

In this work the algorithm is developed further, to address the issues unresolved in the previous publication. It extends the decomposition to handle all types of EVG entities, providing a complete decomposition of the volume into regions, such that each region contains at most one “interior” face of the EVG. Thus it can be viewed as using all the structural (symbolic) information of the medial axis.

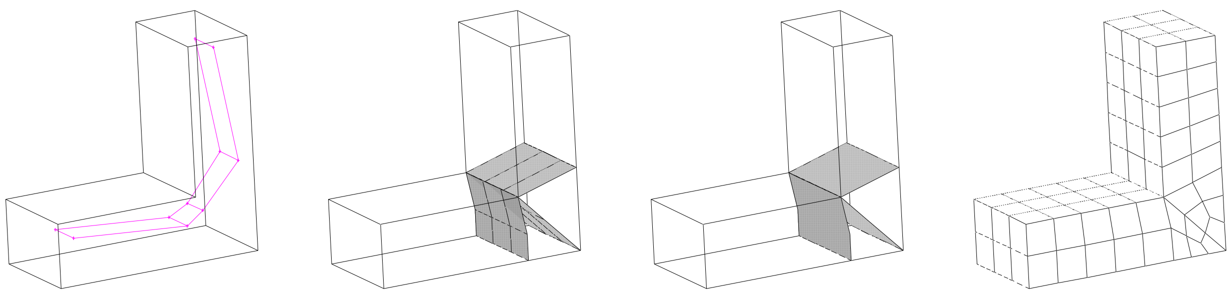
The paper analyses the types of sub-volumes that are generated, based on the EVG faces contained, and establishes the meshing techniques applicable to them. The issues of conformity between the mesh of adjacent sub-volumes and of mesh quality are addressed. Further post-processing of the decomposition results to improve the mesh quality is suggested.

The EVG computation and analysis are expanded to non-linear objects, enabling the meshing of non-polyhedral volumes.

The paper is organized as follows. Section 2 reviews the decomposition approach as suggested in [1]. Section 3 defines the decomposition of the volume based on the EVG entities. In Section 4 the structure of the resulting sub-volumes is analyzed and the suitable meshing techniques are discussed. The application of the algorithm is demonstrated on several examples in Section 5. Section 6 discusses the advantages and drawbacks of the algorithm and suggests topics for future work.

2 Algorithm Overview

In this section a general overview of the meshing algorithm is presented. The algorithm consists of three main stages: (1) construction of the EVG of the object, (2) decomposition of the object into simple parts, and (3) actual meshing. The stages are demonstrated in Figure 1.



(a) (b) (c) (d)

Figure 1: The stages of the meshing algorithm applied on an ‘L’ shaped volume: (a) the embedded Voronoi graph (only interior entities are shown); (b) the decomposition faces, generated by the projection of Voronoi edges and vertices; (c) The decomposition of the volume into four sub-volumes, showing the decomposition faces after they were merged; (d) the resulting volume mesh (before smoothing).

2.1 The Embedded Voronoi Graph

In the first stage of the algorithm, an EVG of the input volume is computed.

Terminology: Let V be a volume. The *entities* of V are the vertices, edges and faces of V. An entity a is *incident* on an entity b if (1) a is an end vertex of the edge b, or (2) a is a vertex of the face b, or (3) a is a bounding edge of the face b. Two entities a and b are *adjacent* if they are not incident one on the other, and there is another entity incident on both of them. For example a face and an edge that share a vertex. Two entities are *disjoint* if they are not adjacent and are not incident one on the other. A Voronoi region R_a is the locus of points that are closer to entity a than to any other entity of V. The boundaries of the Voronoi regions comprise the *Voronoi diagram* of V, $VD(V)$. $VD(V)$ is comprised of *Voronoi faces*, *Voronoi edges*, and *Voronoi vertices*. The *governors* of a Voronoi element are its nearest entities. In the non-degenerate diagram, a Voronoi face has two governors, a Voronoi edge has three governors, and a Voronoi vertex has four governors. In a degenerate diagram, edges and vertices can possess more governors. A Voronoi entity is viewed as *interior* if it does not contain points on the volume boundary. A Voronoi face is interior if its governors are disjoint and a Voronoi edge or Vertex are interior if they have at least one pair of governors which are disjoint. When the faces of the volume are linear, its medial axis can be easily obtained from the Voronoi diagram by deleting Voronoi faces and edges leading to concave vertices and edges of the volume. Constructing the medial axis from the Voronoi diagram is more problematic when the faces of the volume are not linear, since in this case the medial axis might contain elements governed by a single entity. Such elements do not belong to the Voronoi diagram.

Etzion et al [20] defined the **Embedded Voronoi Graph** (EVG) that approximates the Voronoi diagram of a volume, and provided a simple algorithm to construct it. The EVG contains all the symbolic information present in the Voronoi diagram and provides the geometric approximation of specific elements of $VD(V)$. The procedure used to construct the EVG allows obtaining the full symbolic structure of the Voronoi diagram and then computing *locally* the approximation of the diagram entities of interest. The algorithm for computing the EVG is easy to implement and robust. Its convergence and correctness are proven in [21].

The basic algorithm for constructing the embedded Voronoi graph as presented in [20], operates only on polyhedral volumes. In [21] it was extended to handle non-polyhedral volumes as well, thus allowing the use of EVG for decomposition and meshing of any type of volume.

The decomposition algorithm as defined below is based on the symbolic information of the Voronoi diagram (EVG) and the geometric information is required only when the actual decomposition surfaces are created. The exact location of the Voronoi element is of little importance, since those surfaces are not part of the final mesh; their only role is to decompose the volume into simple parts. Thus, the EVG is suitable for our purposes, and there is no need to compute the exact Voronoi diagram, which is much more difficult.

In Figure 1(a) an EVG of an 'L' shaped volume is displayed. The EVG displayed in this figure is not symmetric since it gives an *approximation* to the location of the Voronoi elements. This approximation is sufficient for the meshing algorithm. However a finer approximation can be computed if needed.

2.2 Object decomposition

This is the major step of the algorithm. The main observation on which this stage is based is as follows. Consider a Voronoi face f governed by two entities a and b of V. The sub-volume V_{ab} defined by a, b, and the projection of f to a and b, has the following attractive properties:

- V_{ab} is wholly governed by a and b [20], and therefore does not intersect any other entity of the volume nor any other element of its Voronoi diagram.
- V_{ab} does not intersect any other sub-volume defined similarly by two entities of the object. Therefore V_{ab} can be meshed independently of other sub-volumes.

Thus the embedded Voronoi graph of V provides a natural decomposition of the volume into a set of well defined and disjoint sub-volumes V_{ab}.

Moreover if a and b are non adjacent faces of the volume then V_{ab} is a sweep from a to b, hence can be meshed by a standard method to mesh sweep volumes. Such Voronoi faces are denoted by f_s and the resulting sub-volumes by V_s (Figure 2).

In [1] the decomposition of the volume was limited to separating the V_s type volumes. This was achieved by projecting Voronoi edges that bound interior faces on their respective governors, and creating decomposition faces between the edges and their projections. Such decomposition produces a set of meshable sub-volumes for many volume shapes (eg. Figure 17). However, as shown in Figure 18(b), it may not be sufficient for a general volume. In Section 3 the decomposition is extended to generate sub-volumes governed by other types of interior Voronoi faces, thus providing a complete decomposition of the volume into a set of meshable parts.

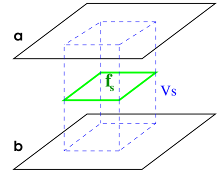


Figure 2: A V_s type sub-volume

Whether the decomposition is limited to f_s type Voronoi faces, or handles all types of interior Voronoi faces as in this work, it is achieved by projecting the Voronoi edges to their governors. To create a disjoint set of sub-volumes, the decomposition at the Voronoi vertices needs to be defined so that the decompositions induced by the edges which meet at the vertex will be connected correctly. Figure 1(b) shows the decomposition faces, resulting from the projection for the 'L' shaped volume.

The decomposition is performed using virtual topology operators [22], avoiding the complex computations required for actual geometric decomposition and allowing the easy removal of the decomposition surfaces later at the smoothing stage. Another approach would be subdivide the volume mesh directly by creating mesh boundaries like the "ribs" in [17].

2.3 Meshing

The volume mesh is constructed by meshing the set of sub-volumes produced by the decomposition (Figure 1(d)). In Section 4 the meshing procedures suitable for the different sub-volume types are discussed.

When meshing the sub-volumes, mesh conformity has to be maintained between the adjacent sub-volumes. When all the sub-volumes can be meshed by sweep between original volume faces, map, or submap this can be achieved by using an interval assignment algorithm on the set of the sub-volume faces [23]. In other cases, there might be a need for a more complex procedure such as use of integer programming [16], as explained in Section 4.

After the sub-volumes are meshed, the basic problem of generating a hexahedral mesh of the volume is solved. However, the restrictions imposed on the mesh by the geometry of the decomposition faces may affect the mesh quality. The fact that the computation of the Voronoi diagram geometry is only approximate, also comes into account, resulting sometimes in non-intuitive positioning of the decomposition, and often in non-symmetric decomposition of symmetric volumes (Figure 1(d)). Since the constraints imposed on the mesh by the decomposition are artificial from the user point of view, removing them can significantly improve the mesh quality.

This can be done either by reassigning the mesh back to the original volume or simply merging all the sub-volumes into one. After the original topology is restored, a smoothing procedure can be applied on the mesh of the faces and the volume improving the mesh quality. There is a variety of techniques for performing the smoothing as reviewed in [2].

The decomposition procedure and the meshing of the resulting sub-volumes are described in detail in the following sections.

3 Volume Decomposition

After the embedded Voronoi graph of the volume is constructed, the volume is decomposed into a set of meshable sub-volumes, by projecting Voronoi edges on their respective governors and creating decomposition faces between the edges and their projections.

In order to decompose the volume into a set of disjoint sub-volumes, each containing at the most a single interior Voronoi face, it is sufficient to do the decomposition only along interior Voronoi edges, while taking care at the vertices to extend the decomposition towards the volume boundary when necessary.

Below we discuss how the edges and vertices are projected, and define the necessary extension of the decomposition faces to the volume boundaries, to generate a set of disjoint sub-volumes.

3.1 Edge Projection

In order to create the volumes, Voronoi edges are projected on their governing entities in V. For each edge the decision of whether to project it on a specific governor and how to do this projection depends on the type of the governor entity and on the relations between the governors. Based on the decomposition goal of creating sub-volumes with one interior Voronoi face or none, the decomposition should separate each region V_{ab} containing an interior Voronoi face from the rest of the volume.

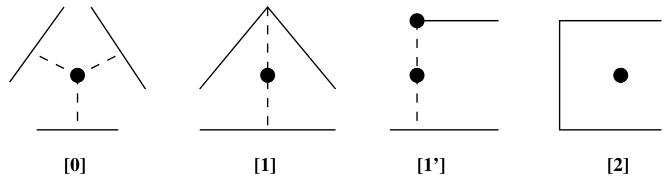


Figure 3: Different Voronoi edges and their projections. The entities of the volume are shown in bold. The decomposition faces are shown as dashed lines. The edge numbering ([0]-[2]) stands for the number of governor entities adjacencies.

The decision on projecting the edge to the governor entities depends on the adjacency relationships between the governors. A pair of edge governors a and b define an interior Voronoi face only of they are disjoint. Thus if all the edge governors are disjoint, to separate the regions the edge needs to be projected to all its governors. If there is only one or no disjoint pairs of governors, then no projection is needed to separate regions. If a pair of governors are adjacent then the edge can be projected to their common boundary. If governor a is incident on governor b then projecting the edge to a will suffice to separate regions governed by a from those governed by b.

This observation defines the projections sufficient to separate the regions of the edge governors. For a non degenerate edge (edge governed by 3 entities of V) it results in a final set of decompositions shown in Figure 3.

Figures 3 and 4 use the following classification for non-degenerate edges introduced in [11] (the numbers correspond to the number of governor adjacencies):

[0] An edge with disjoint governors.

[1] An edge with one pair of adjacent governors. Here a distinction is made as to one of the governors being incident on the other (Figure 3 [1']). A distinction is also made between the case [1] where the angle along the shared entity is not sharp, and the case [1s] where the angle is sharp (Figure 4[1s]).

[2] And edge with three governors, where one governor is adjacent to two others, but they are not adjacent between them.

[3] An edge with each governor having two adjacent governors.

[C] An edge whose governors share a common vertex. Such edge is a part of the Voronoi diagram, but is not interior, since it will necessarily contain the common vertex.

The set of decompositions defined for edges of types [0]-[2] and its extension to the degenerate edges is sufficient to separate the regions of the edge governors. However in order to match the decomposition implied by the edges at common vertices and to limit the set of the generated sub-volume types, sometimes the edge needs to be projected to the governors even if they are adjacent. In Figure 4 we show the two such cases for a non-degenerate edge. In Figure 4 [3] all three governing entities are adjacent (in pairs). Here the decomposition ensures that the sub-volumes are quadrilateral in the edge direction, and guarantees the compatibility of the edge projections at its

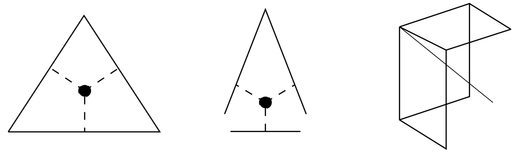


Figure 4: Special Voronoi edges and their projections. [3] Edge with governor entities which are adjacent in pairs. [1s] Edge with two adjacent governors, with a sharp angle between them. [C] A Voronoi diagram edge, which is not interior (the edge is shown as the thin line starting in the corner).

end vertices. In Figure 4 [1s] a pair of the edge governors is adjacent. Projection of the edge to the governors themselves, gives better mesh quality when the angle between the adjacent governors is sharp. It is also sometimes required to achieve the projections compatibility at the end vertices.

The same strategy is used for degenerate edges (i.e. edges with more than three governor entities), with the projections based on the adjacency relations between each pair of governors. Two examples of a degenerate Voronoi edge projection are in Figure 5.

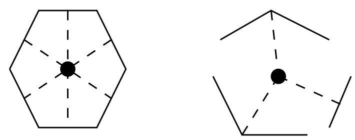


Figure 5: Degenerate Voronoi edges and their projections. Left - edge with each governor adjacent to two others. Right - edge with two adjacent pairs of governors, and one stand alone governor.

The shape of the decomposition face created by the projection depends on the type of the projection *target*, where as explained the target is either the governor or the common boundary between governors. For face governors the projection gives a quadrilateral face, defined by the end vertices of the Voronoi edge and their projections on the governor face. For a vertex target, the projection creates a triangular face defined by the edge end vertices and the target itself. For an edge target, the decomposition face can be either triangular or quadrilateral, based on the type of the governor relations and the Voronoi edge geometry. If the projections of the end vertices on the edge target coincide then the decomposition face is triangular, if not it is quadrilateral.

3.2 Handling of Vertices

In order to decompose the volume along the interior Voronoi edges as described above, the decomposition needs to be defined at the end vertices of the edges. This requires defining the treatment of all the interior Voronoi vertices.

The vertex treatment includes projecting the vertex on its governor entities, based on the desired projection of the vertex edges on the appropriate governor. It also includes extending the decomposition of the edges towards the volume boundary at some of the vertices, to avoid gaps and insure proper separation between the sub-volumes.

In order to form disjoint sub-volumes using the decompositions along the edges, the vertex projections to a governor induced by the edge projections have to be identical. Otherwise there will be gaps in the decomposition boundaries. This restriction is not trivial, since as explained above in may cases the edges are projected not to the governors but to the common boundaries between them.

Hence for each governor of the vertex, the projections to it as defined by each interior Voronoi edge starting at the vertex, need to be checked. Such a check can result in one of three cases, as demonstrated in Figure 6.

- The projection to the governor entities, implied by the edge projections is identical. Eg. (Figure 6(a)), for a vertex with four disjoint governors, for each edge the projection target for the governor is the governor itself.

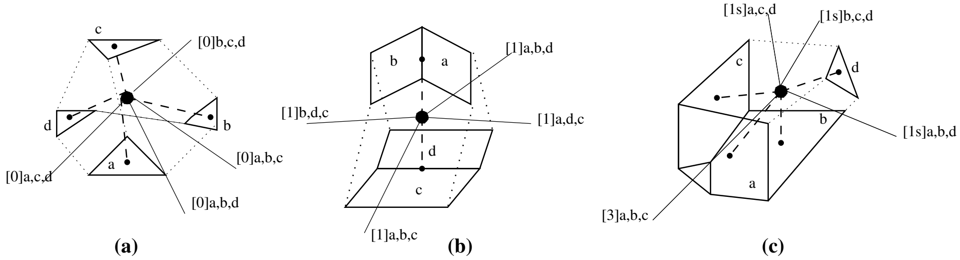


Figure 6: Projection of vertices. The entities of the volume are shown in bold. The edges are shown as thin lines, and the vertex projections are shown as dashed lines. In (a) the projections implied by the edges are identical. In (b) the projections implied by the edges are adjusted to match. In (c) the edge projections are changed so they can match (from [1] to [1s]).

- The projections to the governor entities, implied by the edges are not identical but can be united. Eg. (Figure 6(b)), for a vertex with governors a, b, c, d where a and b are adjacent (sharing $\overline{ab}$) and c and d are adjacent (sharing $\overline{cd}$). Here the vertex has four edges of type [1], governed by (a, b, c), (a, b, d), (a, c, d) and (b, c, d). For each the implied projection for the two adjacent governors, is to the boundary between the governors, and for the third governor to itself. Those projections can be united if the vertex is projected to $\overline{ab}$ for governors a and b, and to $\overline{cd}$ for c and d.

- The projections to the governor entities, implied by the edges can not be united. Eg. (Figure 6(c)), for a vertex with governors a, b, c, d where a and b are adjacent (sharing $\overline{ab}$), b and c are adjacent (sharing $\overline{bc}$), c and a are adjacent (sharing $\overline{ca}$), d is disjoint from all others, and $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ are disjoint as well. Hence, edges (a, b, d), (b, c, d) and (a, c, d) are of type [1] and edge (a, b, c) is of type [3]. Using the projections implied by the edges creates inconsistencies, as for example for governor a the projection implied by the edge (a, b, d) is to $\overline{ab}$, and by the edge (a, c, d) is to $\overline{ac}$. In this case the type of projection for the edges needs to be changed, using [1s] for the edges (a, b, d), (b, c, d) and (a, c, d). Note that changing the type of the edge projection, requires to recheck the other vertex of the edge, since conflicts may arise there. The type changing process is finite, and the projections can always be united since making each governor, the projection target for itself, resolves all inconsistencies.

When handling Voronoi vertices, it is not sufficient to define the projection of the vertex to the governors, but it is also required sometimes to extend the decomposition implied by the edges towards the volume boundary to insure proper separation between the sub-volumes.

This extension is required when not all the edges starting at the vertex are projected to their governors. This happens when some of the edges starting at the vertex are of the type [2] or [C] (Figure 4).

While a Voronoi edge of type [2] is incident on only one interior Voronoi face, an end vertex of such an edge may be incident on other interior faces, hence additional decomposition at the vertex might be required to separate the regions governed by those faces.

If such faces exist then the decomposition is done as follows. Suppose that an edge of type [2] is governed by a, b, c where a is adjacent to b and c, and at the Voronoi vertex besides the interior face b, c there is also an interior face b, d. Then, in order to separate the region of the Voronoi face governed by b and c from the rest of the volume, additional decomposition faces are generated between a and b, and between a and c. A decomposition face between a and b is defined by: the Voronoi vertex, the projection targets of the vertex on a and on b (if targets were set by other edges, they are used, if not the projection is to a and b themselves), and the projection to $\overline{ab}$ (the shared boundary of a and b). The decomposition face for a and c is created identically.

In Figure 7 two examples of vertices with [2] type edges are shown. In 7(a) the vertex has only one interior Voronoi face attached and hence no decomposition is required at it. In 7(b) there are two interior Voronoi faces at the vertex and hence the decomposition at the vertex contains in addition to the decomposition given by the [1] type edge also the decomposition faces resulting from the [2] edge treatment.

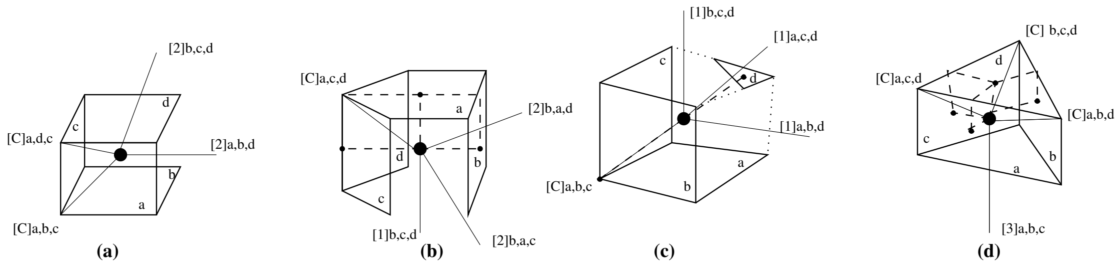


Figure 7: Treatment of vertices with [2] and [C] type edges. The entities of the volume are shown in bold. The edges are shown as thin lines. The vertex projections and the additional decomposition faces are shown as dashed lines. In (a) the vertex has two [2] type edges and two [C] edge. It has only one interior face b, d attached, and hence no decomposition is needed at it. In (b) the vertex has a [1] type edge separating the faces governed by b, c and b, d and hence the treatment of the [2] edges results in addition of decomposition faces between b and a, and $\overline{dc}$ and a. In (c) the vertex has three [1] type edges and a [C] edge. Uniting the projections of the [1] type edges, leads to the projection target being the volume end vertex of the [C] edge. Hence no gaps are created and no extra treatment is needed for the [C] edge. In (d) the vertex has a [3] type edge, which is projected to its three governors and three [C] type edges, each sharing two of those governors. Decomposition faces are added at the [C] edges to prevent gaps.

Voronoi edges of type [C] may also require special treatment at their end vertices. [C] edges are not interior, and hence one of their end vertices correspond to a vertex of the volume. If at a Voronoi vertex v there is a type [C] edge e_c and the vertex is projected (based on other edges starting at it) to one of the governors of e_c, then unless this projection target is the shared volume vertex of e_c (as in Figure 7(c)), a gap will be created between the projection and the volume boundary. The gaps are closed by creating decomposition faces between vertex governors as demonstrated in Figure 7(d).

The considerations of having a single projection for each governor, and of extending the decomposition to the volume boundary at [C] and [2] edges when necessary are not limited to vertices with four governors and are directly applicable to degenerate vertices, both those with more then four edges or with degenerate edges starting at them.

3.3 The Actual Decomposition

After the decomposition faces at the Voronoi vertices and edges are created, the original volume needs to be split into a set of volumes using those faces. The procedure used to create the decomposition faces based on the Voronoi edges and vertices, creates a large number of faces, which can be topologically merged (i.e. have no other faces starting at their common edges). Hence, before splitting the volume, such groups of decomposition faces are merged, removing unnecessary constraints imposed on the mesh. This step is demonstrated in Figure 1(b) to (c).

Thus the actual decomposition procedure consists of three stages: merging decomposition faces, imprinting the faces on the volume boundary and splitting the volume with the faces.

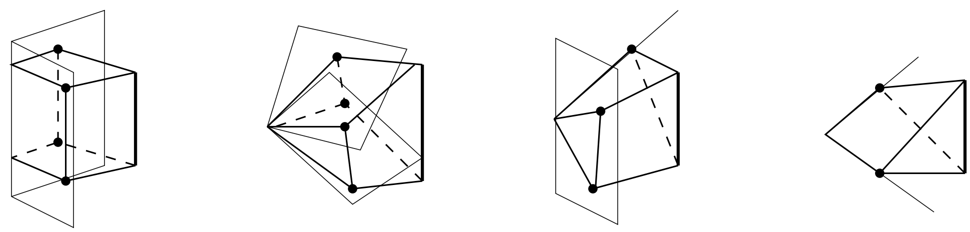
In this work virtual topology operators [22], were used both when constructing the decomposition faces and when performing the actual volume split as explained in [1]. The use of virtual operators minimizes the need for geometric computations during the decomposition. Other tools like construction of mesh element boundaries, like the “ribs” in [17], can be used as well.

4 Sub-Volumes Mesh

The mesh of the original volume is created by meshing the set of sub-volumes resulting from the decomposition, while maintaining mesh conformity along their shared boundaries. In this section the types of the sub-volumes resulting from the decomposition and the meshing techniques applicable to them are described.

4.1 Sub-Volume Classification

Most of the sub-volumes resulting from the decomposition procedure are volumes formed by an interior Voronoi face and the projection of its boundaries to the two face governors. Due to the decomposition used there is no sub-volume containing more then one interior Voronoi face. There are only two cases when sub-volumes do not contain such a face.



(a) **(b)** **(c)** **(d)**

Figure 8: Sub-volumes formed by projecting Voronoi edges to adjacent governors. (a) Voronoi edge governed by two faces sharing an edge. (b) Voronoi edge governed by two faces sharing a vertex. (c) Voronoi edge governed by face and edge, sharing a common vertex. (d) Voronoi edge governed by two edges.

- Sub-volumes generated when a Voronoi edge is projected to two adjacent governors as in Figure 4 [3] and [1s] or Figure 5(a). In Figure 8 the shapes of the sub-volumes generated in this case are shown. Note that this case can exist only for two governors which are adjacent (and not when one is incident on the other). The shape of the sub-volume will depend on both the shape of the decomposition faces and the governor entities. For an edge governed by two faces, which share and edge, when the governor faces involved are convex (or convex in the area of the decomposition), the sub-volume is mappable. For an edge governed by two faces, which share a vertex (if the faces are convex), the sub volume can be meshed as shown in Figure 10(a,b). For an edge governed by an edge and a face, (if the face is convex), the sub-volume is a triangular prism and can be meshed by sweep. For an edge governed by two edges (Figure 8(c)), the volume can be meshed by a template as shown in Figure 10(c,d). However, when the governor faces are non convex in the area of decomposition, the resulting sub-volumes can be too complex, for meshing by map or templates (Figure 11(a,b)).

- Sub-volumes generated when handling [C] type edges at vertices. The sub-volumes in this case are formed by a Voronoi vertex and its projections to three governors sharing a [C] edge (and hence meeting at a shared volume vertex). A sub-volume is created only if the projections to at least three of the governors differ (each pair of different targets forms a decomposition face). The possible shapes of volumes creates for a [C] vertex with three target entities are listed in Figure 9. The treatment of a face target which shares only the vertex with other targets is identical to treatment of similar edge (Figure 9 (c,d)). Note, that if the shared vertex is a governor of the Voronoi vertex, then the vertex will be projected only to it, and no sub-volume will be created. The common cases of sub-volumes created at [C] edges are Figure 9(a) and (b), which can be meshed by map and sweep respectively. For Figure 9 (c,d) the template meshes shown in Figure 10(e,f) and (g,h) respectively can be used.

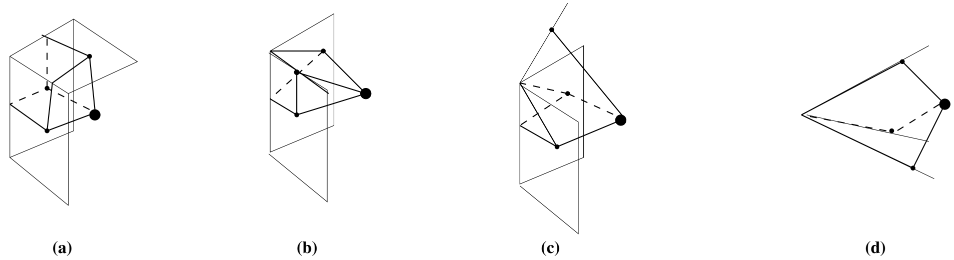


Figure 9: Sub-volumes formed at a Voronoi vertex with [C] type edge. (a) The target entities are faces which share edges. (b) The target entities are two faces which share an edge, and an edge incident on one of the faces. (c) The target entities are a face and two edges, not incident on it. (d) The target entities are three edges.

In all other cases the sub-volume will contain a single interior Voronoi face. The shape of the volume and the meshing algorithm applicable will be defined based on the type of governors and the shape of the Voronoi face.

In real life volumes most sub-volumes will be governed by a pair of governors which includes a face (face & face, face & edge, face & vertex), with other sub-volumes being both rare, and when existing relatively small compared to the original volume size. The reason is that in order for a region to have no face governors it has to be highly concave. Moreover, a convex region of a volume is governed by faces only, and hence will contain only sub-volumes governed by two faces.

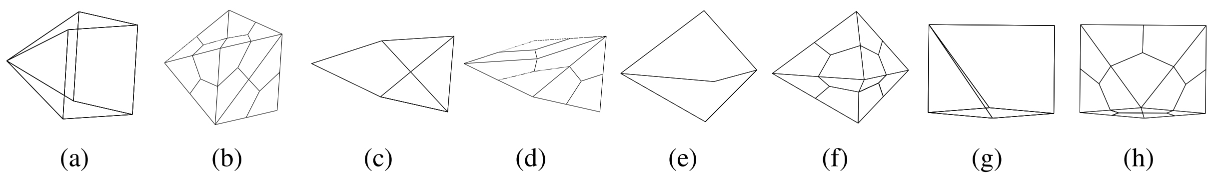
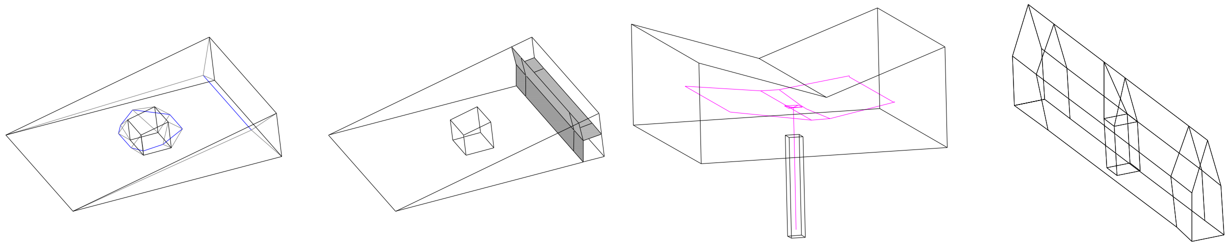


Figure 10: Template meshes of some common sub-volumes. Note that using the template mesh, prevents sweeping along the quad faces in the adjacent volumes. (a,b) A sub-volume generated by projecting a Voronoi edge to two governor faces sharing a vertex. (c,d) A volume shaped as a union of two tetrahedra (generated by projecting a Voronoi edge to two adjacent governor edges). (e,f) (g,h) Two types of sub volumes generated at [C] vertices, with edge governors.

Sub-volume governed by two faces: This is the basic and most common type of sub-volumes. For such sub-volume the edges that bound it all have the two governor faces a and b as their governors. The sub-volume is formed by either the bounding edge projections on the two governor faces a and b or in case of [2] type edges, directly by the relevant part of the volume boundary.

The projection of the bounding edges on a and b will always give quadrilateral faces. This is a result of the way the edge projections are defined. As explained, the target for edge projection to a governor is either the governor itself or a common boundary between the governors. If the target for projection to a or b is the face itself or an edge incident on it, then the projection creates a quadrilateral face as defined.

The remaining case to be considered is of the projection target being a vertex. Below it is shown that such a target is impossible for a face governor. Consider two governors a and c where a is a face and a and c share a common vertex d. If c is an edge or a face, if the angle between a and c at d is convex, the projection to a and c will be of [1s] type, with a and c serving as the projection targets. If the angle is concave, then d is also a governor (this includes the case of c being the vertex). However, since a region governed by a face and a vertex incident on it is a 1D line, an edge governed by two faces a and b and vertex d incident on a can not exist. Hence a vertex can not be a projection target for a or b.

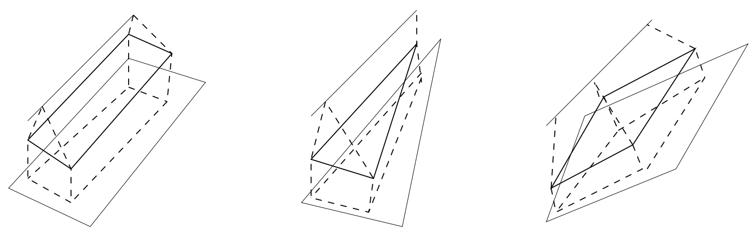


(a) (b) (c) (d)

Figure 11: Examples of sub-volumes unmeshable by sweep or map. (a,b) A sub-volume governed by two non convex adjacent faces: (a) The volume with the embedded Voronoi graph, including [C] type edges. (b) The decomposition at the [3] type edge. The final decomposition, will include decomposition around the through hole as well, however since the Voronoi face governed by the top and bottom faces has a hole, the sub-volume will also have a hole and will not be meshable. (c,d) A sub volume governed by an edge and a face, with a Voronoi face with an interior hole: (c) The volume with the embedded Voronoi graph. (d) The sub-volume created around the Voronoi face governed by the concave edge and the face below (note the hole around the hole of the Voronoi face).

Based on the definition of the decomposition procedure, the only faces (partial faces) of the original volume that can take part in this sub-volume (besides a and b) are faces sharing a [2] type edge with a and b. The treatment of the Voronoi vertices incident on a [2] type edge as described above guarantees that those faces are also quadrilateral and can be swept along between a and b.

Hence all the sub-volume faces, besides a and b are quadrilateral and can be swept along. Since each Voronoi edge is projected to both a and b, the regions of a and b included in the sub-volume have identical topology. So a sub-volume governed by two faces a and b can always be meshed by sweep from a to b. Note that both a and b are exterior faces, hence to maintain conformity with adjacent sub-volumes only the quadrilateral decomposition faces are considered.



(a) (b) (c)

Figure 12: Shapes of sub-volumes governed by face and edge, when the Voronoi face is convex.

Sub-volume governed by face and edge: For this sub-volume the Voronoi edges that bound it have a governor edge e and a governor face f, where e and f are disjoint. Same as for the case of sub-volume governed by two faces, the sub-volume here is formed by either the bounding edge projections on the two governors or in case of [2] type edges, directly by the relevant part of the volume boundary.

Similarly to the case above, the projection of the Voronoi edges to the governor face f creates quadrilateral decomposition faces. The projection of a Voronoi edge to an edge governor, can create either quadrilateral or triangular decomposition faces. A decomposition face is triangular when a Voronoi edge is perpendicular to the governor edge e or when the projection target for e is one of its end vertices.

Consider a [2] type boundary Voronoi edge, governed by e, f and c, where c is adjacent to both e and f. The decomposition faces formed between c and both e and f are quadrilateral, based on the treatment of Voronoi vertices with [2] type edges. Faces formed between c and e will pass through the shared vertex of c and e, thus if c is a face, the sub-volume face formed on it will be triangular.

In case the Voronoi face governed by e and f is convex this analysis shows that the possible shapes of the sub-volumes formed by face and edge governors are as shown in Figure 12. The most common case is a sweep between two triangular faces (Figure 12(a)). In the two other cases (Figure 12(b),(c)) "knife" elements [24] can be used ([24] shows such elements are acceptable for analysis), however it is better to avoid them.

In the general case, when the Voronoi face is non-convex, there can exist sub-volumes formed by face and edge governors which are neither sweepable nor meshable by a final set of template meshes (eg. Figure 11(c,d)), however as can be seen from the example, such cases are rare. If the sub-volume is not a sweep between two triangles it is advisable to merge it with adjacent sub-volumes when possible as explained in Section 4.2.

Sub-volume governed by face and vertex: For this sub-volume the Voronoi edges that bound it have a governor vertex v and a governor face f, where v and f are disjoint. An edge governed by v and f, will have a quadrilateral decomposition face towards the face governor f and a triangular face towards the vertex governor v. There are no [2] type edges governed by v and f, since there is no Voronoi face governed by v and an adjacent edge or face.

If the Voronoi face governed by f and v is convex, the sub-volume generated will be a union of an n sided pyramid and an n sided prism. The two sub-parts can either be meshed separately, by meshing the pyramid first, using a pyramid template, and then using sweep to mesh the prismatic part. Or if there are no other faces attached at some or all the bounding Voronoi edges, the quadrilateral and triangular faces at the edge can be merged, if the resulting sub-volume is meshable by simpler method. If the faces can be merged along all the bounding edges, then the resulting sub-volume will be a pyramid and can be meshed as such. In Figure 15(c,d,e) there is an example of a sub-volume governed by face and vertex which is created by the meshing of the volume in Figure 14. In Figure 15(d) the pyramid and prism parts are meshed separately. In Figure 15(e) the sub-volume is meshed as a single prism after merging two pairs of adjacent triangle and quadrilateral faces.

If the Voronoi face is non-convex but simple (i.e has no interior loops), the sub-part between the Voronoi face and the vertex can be meshed by dividing it first into a set of tetrahedra by triangulating the Voronoi face, and meshing them using the template for a tetrahedral volume. In case the Voronoi face has interior holes (similarly to Figure 11(c,d)), the sub-volume needs to be merged with neighbor sub-volumes, to enable the meshing.

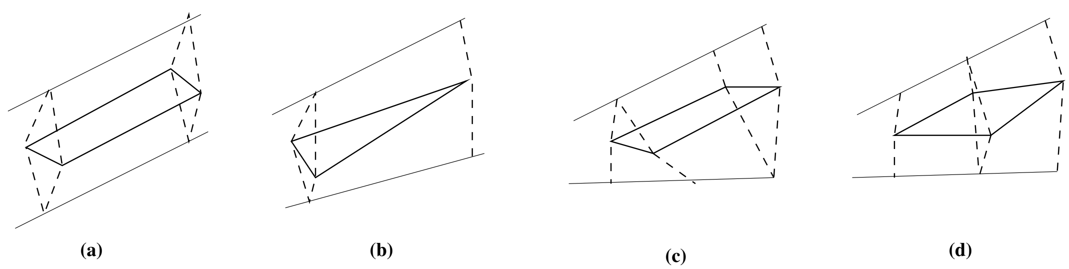


Figure 13: Shapes of sub-volumes governed by two edges.

Sub-volume governed by two edges: The Voronoi edges that bound such sub-volume have two governor edges a and b which are disjoint. Since there are no face governors, the Voronoi face will be simple (with no interior holes) and will follow the shape of the edges. Based on the same considerations as described for a volume governed by face and edge, the sub-regions between the Voronoi face f_{ab} and either a or b have one the shapes listed in Figure 13. For each sub-region same strategy as used for the volume governed by face and edge can be applied, and similarly if a sweep between end faces is not possible, merging the sub-volume with neighbor sub-volumes is advisable to maintain mesh quality.

Sub-volume governed by edge and vertex: The Voronoi edges that bound such sub-volume have a governor vertex v and edge e. Since a decomposition face between a Voronoi edge and a governor vertex is always triangular, the sub-region between the Voronoi face and the vertex v will have an n sided pyramid shape. The region between the Voronoi face and the edge e will have the same shape as in the case of sub-volume governed by two edges, as listed in Figure 13. However, even when this sub-region can be meshed by sweep, since conformity can not be maintained between a sweep and a mesh of an n sided pyramid, mid-point subdivision or other non trivial meshing technique has to be used to mesh such volumes. Hence if possible the sub-volume needs to be merged with adjacent sub-volumes (Section 4.2) to obtain a volume meshable by simple algorithms.

Sub-volume governed by two vertices: The Voronoi edges that bound such sub-volume have two governor vertices. Since a decomposition face between a Voronoi edge and a governor vertex is always triangular, the sub-volume will be a union of two n sided pyramids and can be meshed by a combined template mesh.

4.2 Sub-Volumes Merging

As can be seen in some of the mesh examples displayed in this paper (Figures 1, 14, 18), the use of the EVG or medial axis as basis for decomposition leads to larger number of sub-volumes than would be created by manual decomposition. Using the decomposition technique described here, this number is significantly smaller then in the template based techniques [12], but it can be further reduced.

An attractive approach for reducing the number of sub-volumes is to add a post-processing step to the decomposition procedure described above which will attempt to merge groups of sub-volumes into bigger parts which can still be meshed by basic meshing techniques. This merge procedure, will group adjacent sub-volumes, checking if the formed sub-part can be meshed. The checks for volume suitability for meshing by mapped mesh, submap, sweeping and extended sweeping (like [17]) can be similar to those used in Fluent's Gambit [25] software.

In some cases, as mentioned above, merging the sub-volumes is essential to be able to mesh the volume, like in Figure 11(c,d), or when maintaining mesh conformity is a problem, like when sub-volumes need to be meshed using templates as in Figure 10.

Another reason for merging sub-volumes into bigger sub-parts is cases of sub-volumes on which high mesh quality can not be obtained, and where the merging will significantly improve the mesh quality by removing unnecessary constraint on the mesh. An example of the merge effect on the mesh is shown in Figure 14(c) and (d).

Different ordering of sub-volume check for grouping can lead to different final subdivisions and different mesh structure. Strategies can include:

- Starting with badly shaped sub-volumes, to reduce the number of low-quality mesh elements.
- Starting with the smaller sized sub-volumes, to enable coarser meshing.
- Using a top-down approach, so first all the volume is viewed as one group and a group is subdivided into two if if can't be meshed, using a heuristic to choose the faces to subdivide along.
- Extensive search of all the possible groupings, with a final step of choosing among all the possible subdivisions, based on sub-volume size, quality or number.

Since at each Voronoi edge there are only two adjacent Voronoi faces, each sub-volume has a small number of neighbor volumes, and hence the check of the possible choices is relatively fast.

Examining the different merging strategies is an important topic for further research.

5 Results

Most of the algorithm has been implemented. The embedded Voronoi graph computation is a stand-alone application running under Unix. The volume decomposition and subsequent meshing are implemented using the meshing and editing tools provided by Gambit [25]. The implementation of virtual topology in Gambit was used for the editing and decomposition operations required. Gambit was also used for sweeping and mapped meshing of the sub-parts. The run time of the algorithm is comparable to other mesh methods.

The full implementation of the algorithm is still underway. Currently it lacks the post-processing stage, which includes the procedures to merge sub-volumes into groups prior to meshing, and the procedure for mesh reassignment to the original volume followed by smoothing of the volume mesh as a whole.

The algorithm is demonstrated on several complex real life examples. The example models include many of the possible decomposition types described above. The Voronoi diagrams of the models include degenerate entities, and entities governed by edges and vertices of the volume.

The first example is shown in Figure 14. The volume is non-linear and its Voronoi diagram contains several edges and faces governed by volume edges, as well as a Voronoi face governed by a vertex and a face of the volume. In Figure 15 two sub-volumes governed by pairs of face & edge, and face & vertex are shown together with the mesh generated for them. The subdivision of the volume based on the embedded Voronoi graph contains twenty nine volumes, and the mesh on it is shown in Figure 14(c). After merging the wedge shaped sub-volumes governed by edges in the upper part of the volume, and merging the pairs of volumes at the two cylinder tops, to improve mesh quality, the subdivision contained nineteen volumes, and the mesh on it is shown in Figure 14(d) and (e).

In Figure 16 a hollow pyramid volume is shown. This volume, as well as the one above, can not be meshed by sweep or uni-axial combination of sweeps. Because of the different principal directions of its sub-parts, using a grid based algorithm on it is likely to lead to poor mesh quality. The volume is decomposed into four sweepable sub-volumes, corresponding to the four sides of the pyramid, eight sub-volumes sweepable along the interior hole concave edges and four regions governed by the interior concave vertex and the four outside faces of the volume. The decomposition here is minimal in a sense that that merging any sub-volumes into parts which can still be meshed, will not lead to improvement of the mesh quality.

Figure 17 shows a mesh of a support arm. For this volume decomposition along Voronoi edges with at least two face governors is sufficient to divide it into a set of meshable sub-volumes. The reason for this appears to be that the decomposition of the volume into convex regions is clearly defined with the feature interaction regions being very simple. The volume is divided into seven sub-volumes, and the final mesh contains 1259 element (Figure 17(d)).

In Figure 18 an example is shown of a simple volume (ashtray) where decomposition only along Voronoi edges with two or more face governors is not sufficient. The sub-volumes created by such decomposition are shown in Figure 18(b). The decomposition results in five sub-volumes containing f_s type faces and a sub-volume forming the remainder of the original volume which has a shape of a “table-frame” - a four sided loop with four “legs” attached at its corners. Clearly this sub-volume can not be meshed as a whole by neither map or sweep. Using decomposition along all interior edges we get the subdivision shown in Figure 18(c). Here the subdivision contains five sub-volumes governed by pairs of faces, fourteen sub-volumes governed by edge and face and twelve sub-volumes (at the corners of the ashtray) governed by face and vertex. After merging groups of sub-volumes to improve the mesh quality, the subdivision contains seventeen sub-volumes, and the final mesh contains 1226 elements (Figure 18(d)).

6 Summary

In this paper we presented a hexahedral mesh generation algorithm. The algorithm uses the embedded Voronoi graph of the volume to decompose the volume into simple parts that can be meshed using basic meshing methods.

The approach presented here is general and automatic. It handles any volume, even if its medial axis is degenerate. The embedded Voronoi graph provides complete information regarding proximity and adjacency relationships between the entities of the volume. Hence, decomposition faces are determined unambiguously, without any further geometric computations. The sub-volumes computed by the algorithm are guaranteed to be well-defined and disjoint. The decomposition directions depend on the object and not on an arbitrary external coordinate system, as in grid-based methods.

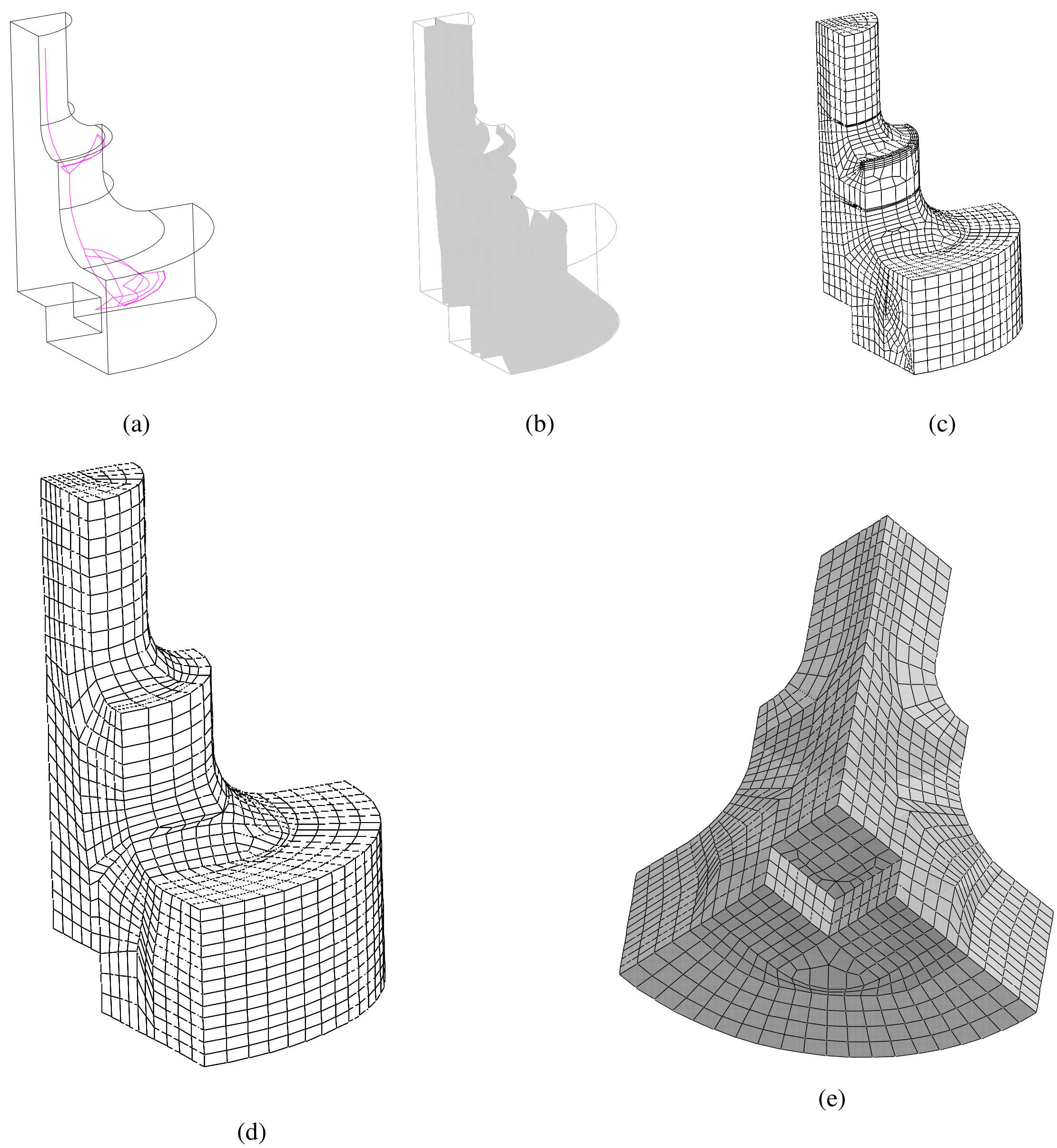


Figure 14: A meshing example: (a) The initial volume with the embedded Voronoi graph. (b) The decomposition faces generated based on the embedded Voronoi graph. (c) Mesh of the set of volumes, generated by the decomposition. (d) Mesh of a set of volumes, after merge of several volumes, to improve mesh quality (front view). The mesh contains 3625 elements. (e) Mesh of a set of volumes after merge (back view).

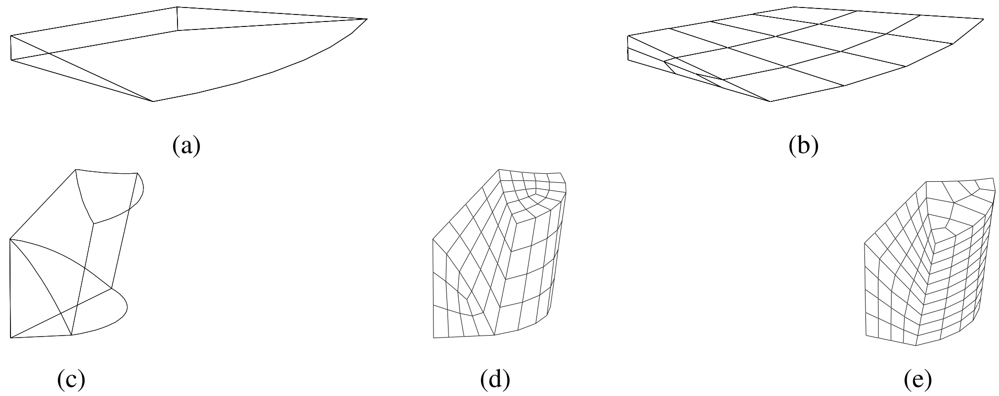


Figure 15: Mesh of two sub-volumes of the volume in Figure 14: (a) A sub-volume created by decomposition at a [3] type Voronoi edge governed by an edge and two faces (at the top of the volume). (b) The mesh of the sub-volume (a triangular prism). (c) A sub-volume governed by a face and vertex (at the center of the volume). (d) A mesh of the sub-volume without merging faces. The mesh consists of a tetrahedron mesh and a mesh of a triangular prism. (e) A mesh of the sub-volume after merging two pairs of side faces. The mesh is done as on a triangular prism.

The decomposition algorithm can use the approximation of the medial axis as given by the EVG and has no need for the exact medial axis geometry as do template based medial axis decomposition methods like [12].

This work is an extension of the decomposition algorithm presented in [1]. The algorithm presented here handles all types of EVG entities and provides a complete decomposition of the volume into sub-volumes, such that each sub-volume contains at most one "interior" face of the EVG The paper provides a full classification of the sub-volume types and the meshing techniques available for them, based on the EVG faces contained.

As demonstrated in the examples, the approach is applicable to a wide range of volumes and provides a good quality mesh.

The work also points out to several drawbacks of the approach presented. A drawback which is inherent to using the EVG or the medial axis is that the medial axis is strongly based on the model geometry and sensible to scaling. Hence, using it can result in over-decomposition of the volume. For example if the 'L' shaped volume in Figure 1 was thiner, it's EVG (medial axis) would contain a single interior face, and it wouldn't have been split into several sub-volumes, and would be meshed as a sweep between the 'L' shaped faces.

This over-decomposition can be avoided by using the sub-volume merging procedure as suggested in Section 4.2. Further research is required to implement the procedure and check the different merging strategies suggested.

During the classification of the sub-volumes (Section 4.1), it was shown that some sub-volumes can not be meshed by basic meshing techniques, maintaining reasonable mesh quality. For most of those it seems that merging with neighbor sub-volumes will result in a meshable volume. A more comprehensive check is required to verify this.

It has been also shown that for some volumes like Figure 11(a,b) the EVG or the medial axis do not provide the decomposition sufficient to mesh the volumes and another meshing technique has to be used.

In this work the meshing algorithm was extended to handle non-polyhedral volumes. The presented algorithm uses the EVG which approximates the Voronoi diagram of the volume and not the medial axis. The Voronoi diagram does not include points that have multiple projection points on a single volume entity. Hence, when using the EVG for decomposition, there will be no decomposition between two parts of a single volume entity (like is often desirable for ball or cylinder faces). Providing an algorithm that will approximate the medial axis, is essential to be able to decompose such volumes.

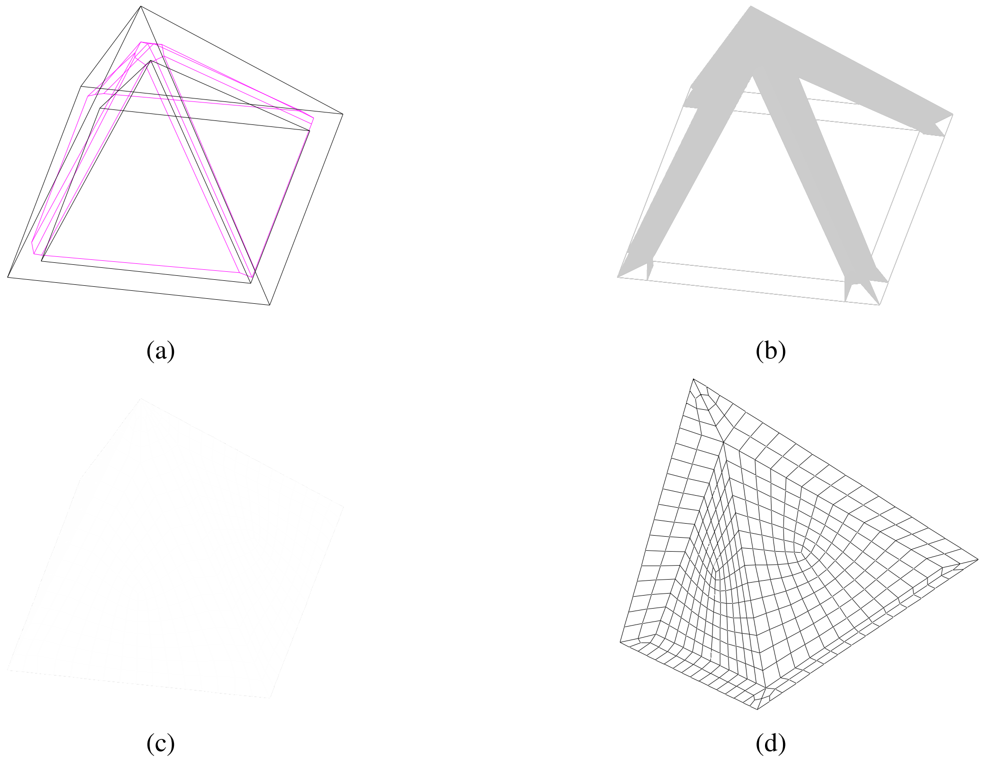


Figure 16: Mesh of a hollow pyramid: (a) The initial volume with the embedded Voronoi graph. (b) The decomposition faces generated based on the embedded Voronoi graph. (c) Mesh of the pyramid (before smoothing). The mesh contains 1472 elements. (d) Mesh of half the pyramid, view from inside.

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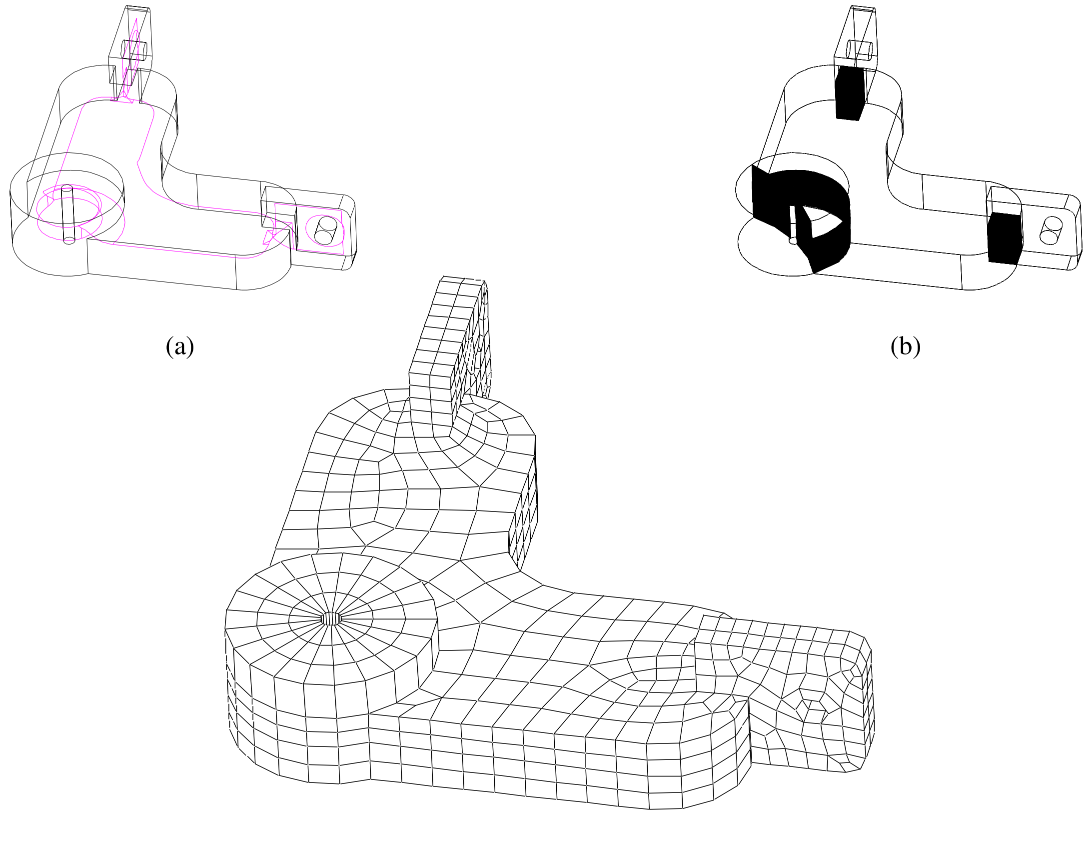

(a) (b)

(c)

Figure 17: Mesh of a support arm: (a) The initial volume with the embedded Voronoi graph. (b) The decomposition faces generated based on the embedded Voronoi graph(only Voronoi edges with two or more face governors were used). (c) Mesh of the arm. It contains 1259 elements.

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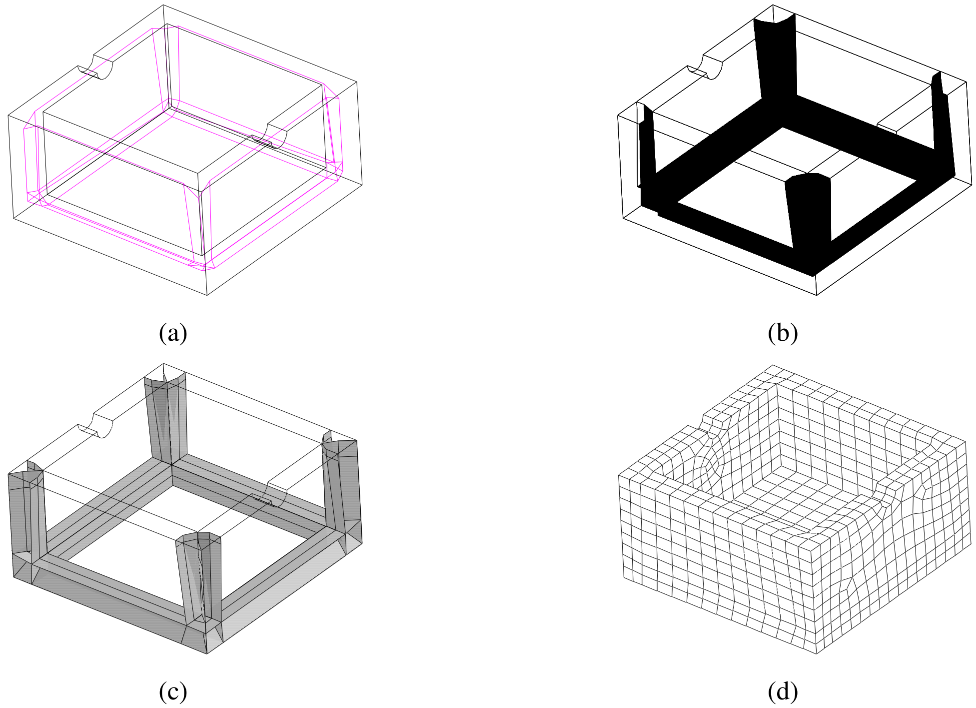

Figure 18: Mesh of an ashtray shape: (a) The initial volume with the embedded Voronoi graph. (b) The decomposition faces generated based on the embedded Voronoi graph, using only Voronoi edges with two or more face governors. (c) The decomposition faces generated based on the embedded Voronoi graph, using all interior Voronoi edges. (d) Mesh of the volume (after merging groups of sub-volumes). The mesh contains 1226 elements.

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