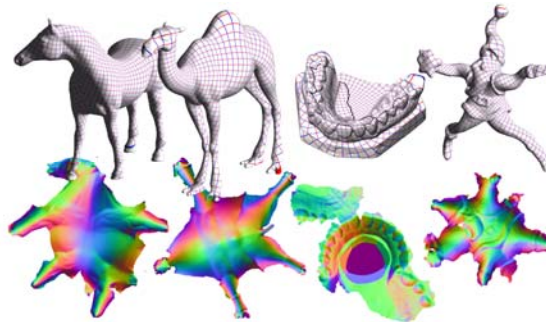
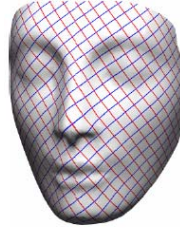
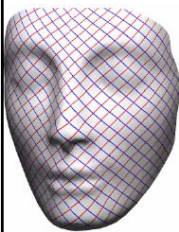


Parameterization – Free Boundary



Free Boundary Methods

- Direct energy minimization
 - Example: Least Squares Conformal Map (LSCM)....
- Indirect
 - Example: Angle Based Flattening (ABF)....



rights [Floater 2003] on a circular domain.

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(LSCM [Lévy et al. 2002].

Free vs Fixed



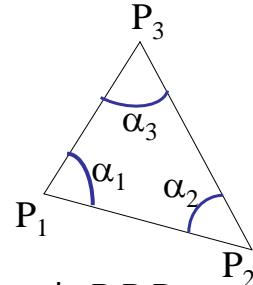
LSCM – Geometric Interpretation

- Algebraic Interpretation:
 - Minimize conformal energy

$$E_C = (\sigma_1 - \sigma_2)^2 / 2$$

- Geometric Interpretation:

- Use triangle similarity
- Given angles $\alpha_1, \alpha_2, \alpha_3$ of a triangle $P_1P_2P_3$ in 2D we have



$$P_3 - P_1 = \frac{\sin \alpha_2}{\sin \alpha_3} R_{\alpha_1} (P_2 - P_1),$$

$$R_\alpha = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$



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LSCM

- In map from 3D to 2D might be impossible to keep angles exactly
 - Use least-squares

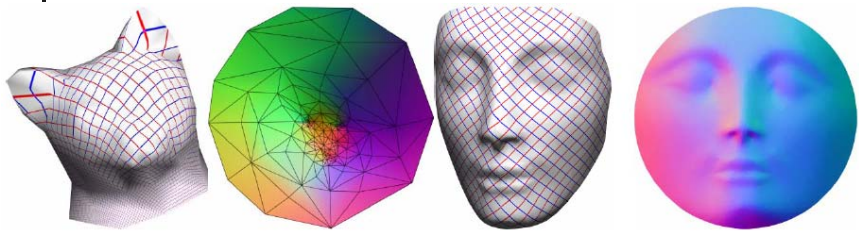
$$\min \sum_i \left(P_3^i - P_1^i - \frac{\sin \alpha_2^i}{\sin \alpha_3^i} R_{\alpha_1^i} (P_2^i - P_1^i) \right)^2$$

- To solve need to fix two vertices
 - Obtain linear system
 - Choice of vertices affects solution
- Can have flips

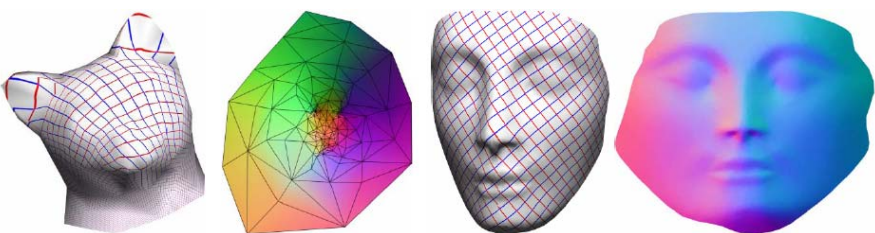


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Examples



Parameterization with mean value weights [Floater 2003] on a circular domain.

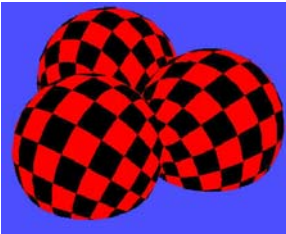
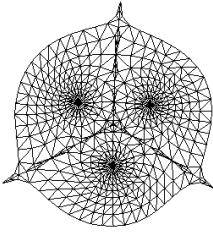


Parameterization with LSCM [Lévy et al. 2002].

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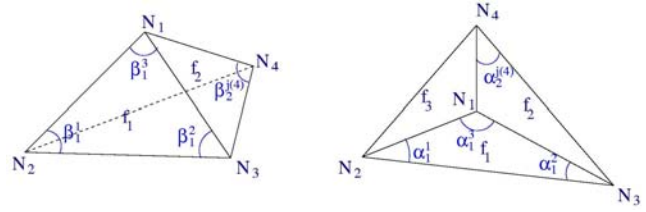
ABF: Angle Based Flattening

- Triangular 2D mesh is defined by its angles
- Formulate parameterization as problem in angle space
- Angle based formulation:
 - Distortion as function of angles (conformality)
 - Validity: set of angle constraints
 - Convert solution to UV

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ABF Formulation

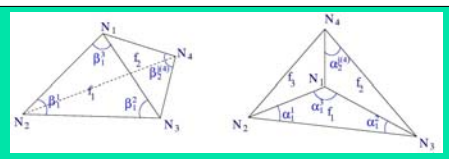


- Distortion:
 - 2D/3D angle difference

$$\sum_{t \in T, j=1..3} w_j^t (\alpha_j^t - \beta_j^t)^2, w_j^t = 1/\beta_j^{t^2}$$



ABF Formulation



- **Constraints:**
 - Triangle validity:

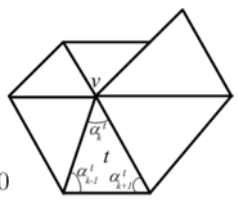
$$\forall t \in T, \alpha_1^t + \alpha_2^t + \alpha_3^t - \pi = 0;$$
 - Planarity:

$$\forall v \in V_{int}, \sum_{(t,k) \in v^*} \alpha_k^t - 2\pi = 0$$
 - Reconstruction

$$\forall v \in V_{int}, \prod_{(t,k) \in v^*} \sin \alpha_{k \oplus 1}^t - \prod_{(t,k) \in v^*} \sin \alpha_{k \ominus 1}^t = 0$$

- Distortion:

$$\sum_{t \in T, j=1..3} w_j^t (\alpha_j^t - \beta_j^t)^2, w_j^t = 1/\beta_j^{t^2}$$



Positivity
 $\alpha_j^t > 0$

- Solve - constrained optimization (Lagrange multipliers)





Angle to UV Conversion

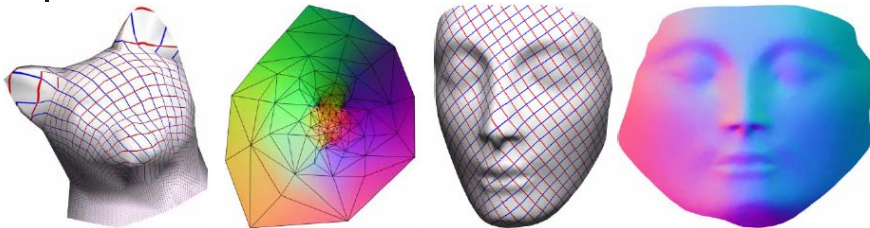
- Alternative 1: Use computed angles as input to LSCM
- Alternative 2: Unfolding
 - Choose one edge & place in 2D (keep length)
 - Based on computed angles place third vertex of triangles sharing this edge in 2D
 - Intersection of two rays
 - Repeat recursively



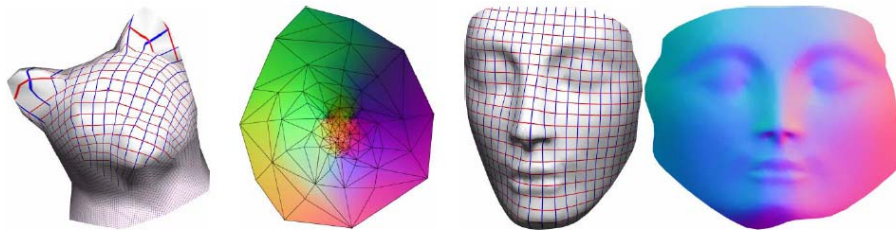
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Examples

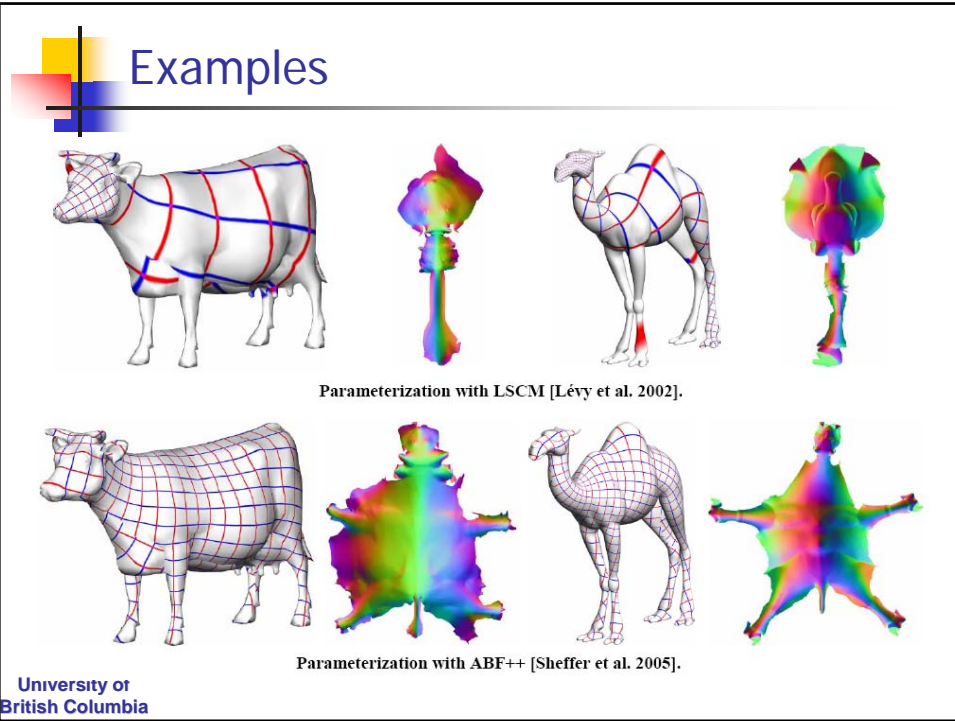


Parameterization with LSCM [Lévy et al. 2002].




Parameterization with ABF++ [Sheffer et al. 2005].

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2D Parameterization Summary


- Needed for many processing operations
- Distortion/Bijection important
- Trade-of quality/efficiency (as always...)
- Very popular topic (50+ major publications in last 6 years)
- More Issues
 - Segmentation/Cutting
 - Constraints



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Numerical Issues



Minimization with Constraints

- Need to
 - Find x such that $F(x)$ minimal
 - WHEN constraints $c(x) = 0$ satisfied
- Achieved when
 - $F'(x) = \mu c'(x)$
 - for unknown μ
- General formulation
 - $F^*(x, \mu) = F(x) + \mu c(x)$
 - Find x, μ which extremize F^*
 - Known as min-max
 - min on x
 - max on μ

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Solution

- Use Lagrange Multipliers

$$F^*(\alpha, \mu) = F(\alpha) + \mu_1 g^2(\alpha) + \mu_2 g^3(\alpha) + \mu_3 g^4(\alpha)$$

- Solve the min-max problem (minimum on α , maximum on μ)
- Reached when all derivatives are zero
- Have non-linear system of equations
- Use Newton method to solve



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Minimization (Unconstrained)

- To find x that minimizes $F(x)$ – find x such that $F'(x)=0$
 - Check if got minimum/maximum/saddle point
 - Note: finds **LOCAL** minimum
- Typically no need for explicit check (assume function does not have maxima/saddles)
- Translate problem into: find x such that $f(x)=0$



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Solving Non-Linear Equations - Newton Method

- Consider Taylor expansion

$$f(x_0 + \epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{1}{2}f''(x_0)\epsilon^2 + \dots$$

- Neglect terms > 1

$$f(x_0 + \epsilon) \approx f(x_0) + f'(x_0)\epsilon.$$

- In 1D

- Set x_0 – initial guess
- While $f(x_i)$ not 0
 - $dx = -f(x_i)/f'(x_i)$
 - $x_{i+1} = x_i + dx$



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Newton Method in nD

- $f(x)$ vector
- $\Delta f(x)$ – matrix
- Set x_1 – initial guess
- While $\|f(x_i)\| > 0$
 - Solve $\Delta f(x_i) dx = -f(x_i)$
 - Solve linear system
 - $x_{i+1} = x_i + dx$



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Solving Linear System

- Solve $Ax=B$ (A $n \times n$ matrix)
- Choice I: Compute A^{-1} $O(n^3)$ TERRIBLY expensive
- Choice II: Iterative (Gauss/Gauss-Seidel)
 - Set x to initial guess
 - Solve one equation at a time
 - $A_i x = B_i$ - consider all x_j ($j \neq i$) as constant and compute x_i
 - $x_i = (b_i - \sum a_{ij} x_j) / a_{ii}$
 - Repeat (for all i) till convergence
 - Works only for a very small set of matrices



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Solving Linear System

- Choice III: LU (or LDL^T) decomposition
 - Compute matrices L & U such that
 - $LU=A$
 - L – lower matrix (has 1's on diagonal & 0's above)
 - U – upper matrix (has 0's below diagonal)
 - Use off-the-shelf algorithm/code
 - Take advantage of sparsity (if applicable)
 - Solve:
 - Solve $Ly=B$ (use Gauss iterations)
 - Works (at each point add ONE variable)
 - Solve $Ux=y$ (use Gauss iterations)
 - Start from $i=n-1$ and go "up"
 - Works (at each point add ONE variable)



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