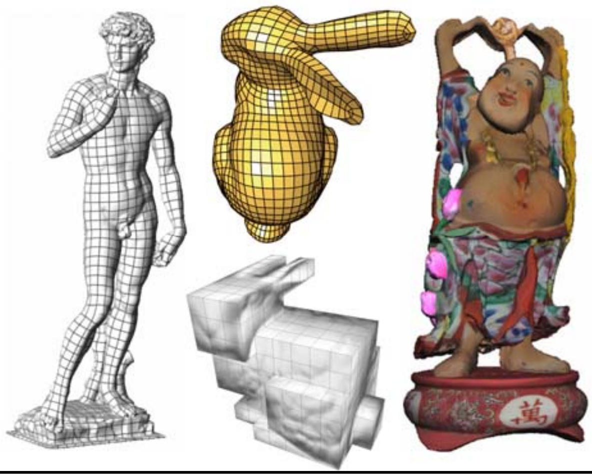
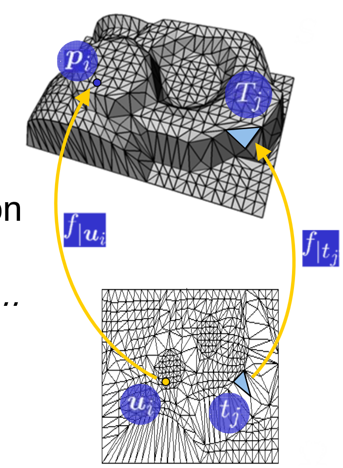
Parameterization - Practice

University of
British Columbia



Mesh Parameterization Methods

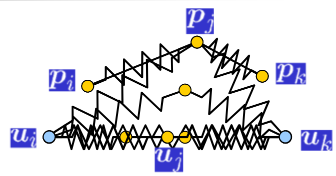
- Fixed Boundary
  - 2D “convex” embeddings
    - Variety of weights
- Free boundary –
  - Direct energy minimization
    - *Example: Least Squares Conformal Map (LSCM)....*
  - Indirect
    - *Example: Angle Based Flattening (ABF)....*



University of
British Columbia

Spring Model

- Replace edges by springs
- Fix boundary vertices on *convex* polygon
- Apply relaxation process
- Energy of spring between p_i and p_j : $\frac{1}{2} D_{ij} s_{ij}^2$
  - Spring constant $D_{ij} > 0$
  - Spring length $s_{ij} = \|u_i - u_j\|$
- Total energy



$$E = \sum_{(i,j)\in\mathcal{E}} \frac{1}{2} D_{ij} \|u_i - u_j\|^2$$

University of British Columbia

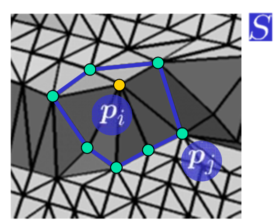
Energy Minimization

- Rewrite

$$E = \frac{1}{2} \sum_{i=1}^{n} \sum_{j\in N_i} \frac{1}{2} D_{ij} \|u_i - u_j\|^2$$

- Partial derivative

$$\frac{\partial E}{\partial u_i} = \sum_{j\in N_i} D_{ij}(u_i - u_j)$$



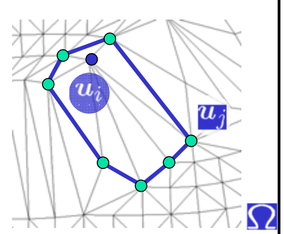
- Achieve minimum (locally) when

Convex Combination

$$u_i = \sum_{j\in N_i} \lambda_{ij} u_j$$

with weights

$$\lambda_{ij} = D_{ij} \Big/ \sum_{k\in N_i} D_{ik}$$



University of British Columbia

Linear System

- Separation of variables

$$\underbrace{u_i - \sum_{j\in N_i, j\le n} \lambda_{ij} u_j}_{\text{unknown parameter points}} = \underbrace{\sum_{j\in N_i, j>n} \lambda_{ij} u_j}_{\text{fixed}} = \bar{u}_i$$

- Linear system

$$\begin{pmatrix} 1 & * & \cdots & -\lambda_{ij} \\ * & 1 & * & \vdots \\ \vdots & * & \ddots & * \\ -\lambda_{ji} & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_n \end{pmatrix}$$

  - Solve separately for u and v

$$\lambda_{ij} = D_{ij} \Big/ \sum_{k\in N_i} D_{ik}$$

University of British Columbia

Why it Works

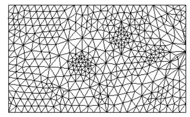
- Theorem [Tutte'63,Floater'01,Maxwel'1864]:
  - If G=<V,E> is a 3-connected planar graph (triangular mesh) then any convex combination embedding ($\lambda_{ij} > 0$) provides bijective parameterization
    - Matrix is
      - sparse
      - diagonally dominant
      - nonsingular

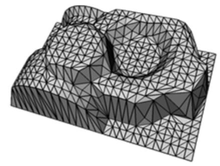
$$\begin{pmatrix} 1 & * & \cdots & -\lambda_{ij} \\ * & 1 & * & \vdots \\ \vdots & * & \ddots & * \\ -\lambda_{ji} & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_n \end{pmatrix}$$

University of British Columbia

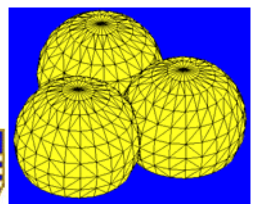
Choice of Weights: Uniform (Tutte)

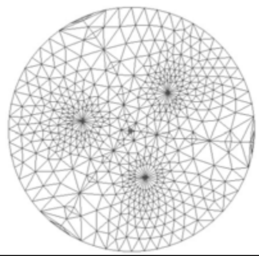
$D_{ij} = 1$ $\lambda_{ij} = \frac{1}{\#N_i}$

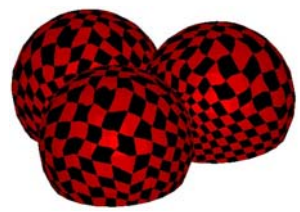




- No shape preservation –equilateral triangles
- Corresponds to basic Laplacian







University of British Columbia

Choice of Weights: Barycentric

- Harmonic/Conformal/Cotan

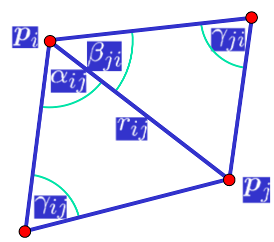
$$w_{ij} = \cot\gamma_{ij} + \cot\gamma_{ji}$$

- Mean-Value

$$w_{ij} = \frac{\tan\frac{\alpha_{ij}}{2} + \tan\frac{\beta_{ji}}{2}}{r_{ij}}$$

- Normalization

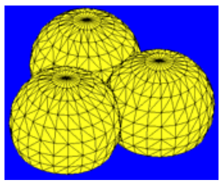
$$\lambda_{ij} = \frac{w_{ij}}{\sum_{k\in N_i} w_{ik}}$$

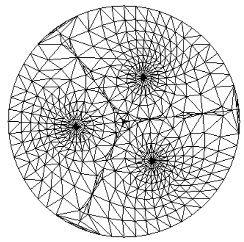


University of British Columbia

Harmonic/Mean-Value Mappings

- Quasi-Conformal

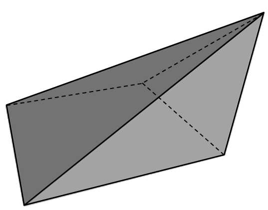


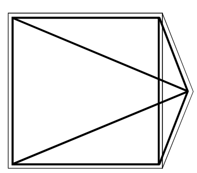


- Linear precision
  - Reproduce planar inputs (same boundary)

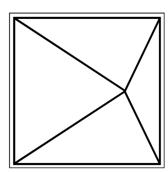
University of British Columbia

Bijectivity (fold-overs)





harmonic

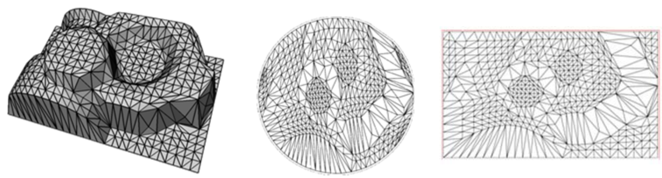


mean value

- Can have fold-overs for negative coordinates
- Mean-value coordinates guaranteed to be positive

University of British Columbia

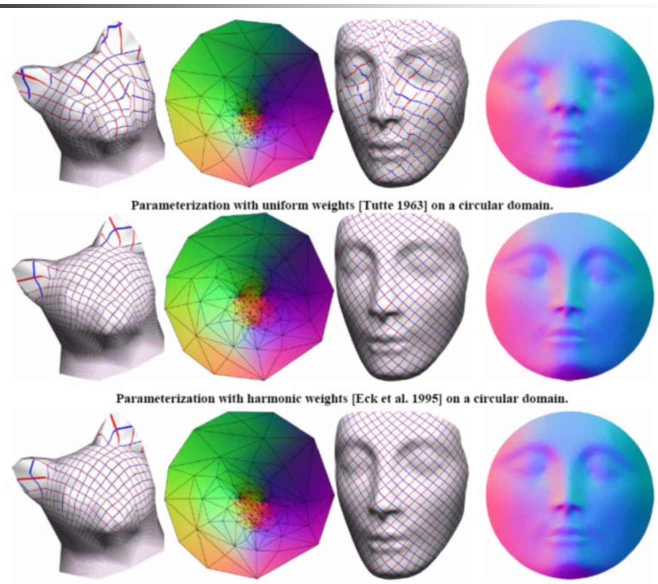
Boundary Mapping



- *Chordal* parameterization around *convex* shape
  - circle
  - rectangle
  - triangle
  - Choice often application specific
    - Reconstruction – rectangle
    - Mapping to base mesh– triangle

University of British Columbia

Examples



Parameterization with uniform weights [Tutte 1963] on a circular domain.

Parameterization with harmonic weights [Eck et al. 1995] on a circular domain.

Parameterization with mean value weights [Floater 2003] on a circular domain.

University of British Columbia