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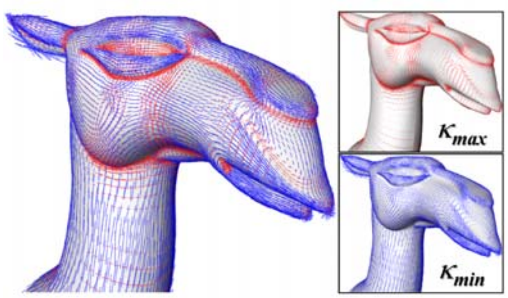
Differential Geometry & Discrete Operators



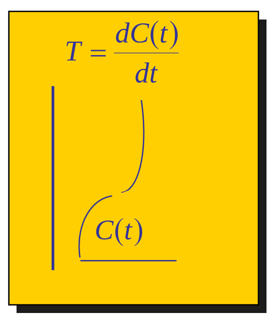
Curves

- Tangent vector to curve $C(t)=(x(t),y(t))$ is

$$T = C'(t) = \frac{dC(t)}{dt} = [x'(t), y'(t)]$$

- Unit length tangent vector

$$\vec{T} = \vec{C}(t) = \frac{[x'(t), y'(t)]}{\sqrt{x'(t)^2 + y'(t)^2}}$$



- Curvature

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

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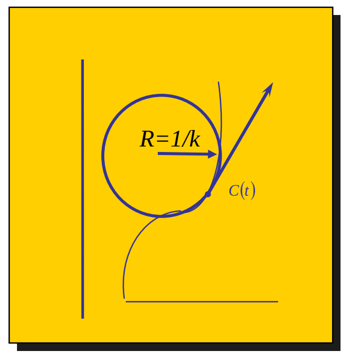
Curvature (Curves)

- Curvature is **independent** of parameterization
  - $C(t)$, $C(t+5)$, $C(2t)$ have same curvature (at corresponding locations)
- Corresponds to radius of osculating circle $R=1/k$
- Measure curve bending



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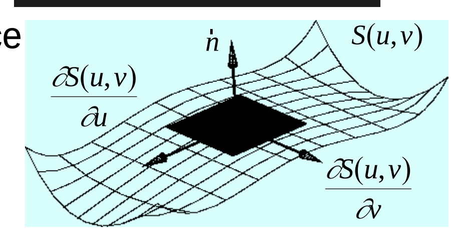
Surfaces

- Tangent plane to surface $S(u,v)$ is spanned by two partials of S:

$$\frac{\partial S(u,v)}{\partial u} \qquad \frac{\partial S(u,v)}{\partial v}$$

- **Normal** to surface

$$\vec{n} = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}$$



  - perpendicular to tangent plane
- Any vector in tangent plane is tangential to $S(u,v)$

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Curvature

- ***Normal curvature*** of surface is defined for each tangential direction

- ***Principal curvatures*** *Kmin* & *Kmax*: maximum and minimum of normal curvature
  - Correspond to two **orthogonal** tangent directions
    - Principal directions
  - Not necessarily partial derivative directions
  - Independent of parameterization

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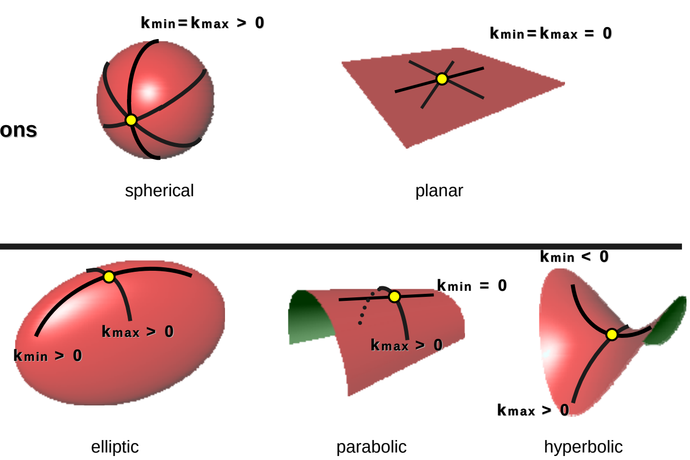
3D Curvature

Isotropic

Equal in all directions

Anisotropic

2 distinct principal directions



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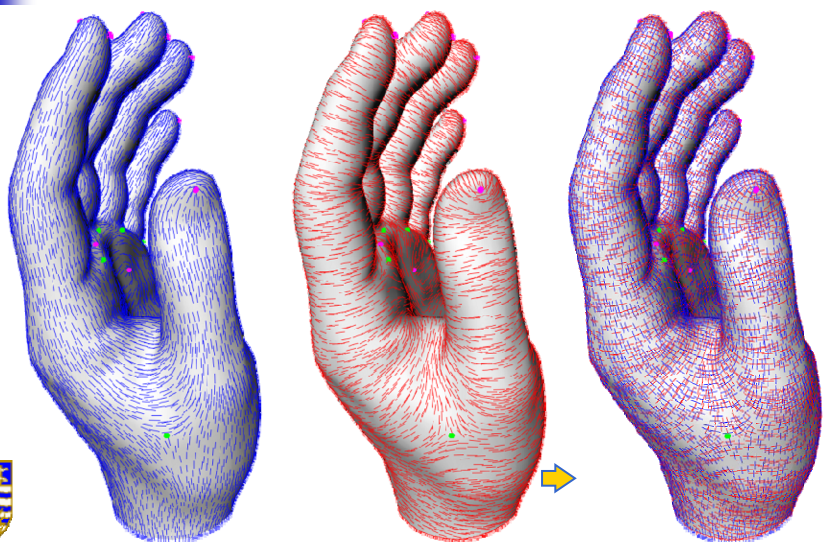
Principal Directions



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min curvature   max curvature

Curvature

- Typical measures:
  - ***Gaussian*** curvature

$$K = k_{min} k_{max}$$

  - ***Mean*** curvature

$$H = \frac{k_{min} + k_{max}}{2}$$

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Curvature on Mesh

- Approximate curvature of (unknown) underlying surface
  - Continuous approximation
    - Approximate the surface & compute continuous differential measures
  - Discrete approximation
    - Approximate differential measures for mesh

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Normal Estimation

- Need surface normal to construct approximate surface
  - Defined for each face
  - Solution 1: average face normals
    - Does not reflect face “influence”
  - Solution 2: weighed average of face normals
    - Weights:
      - Face areas
      - Angles at vertex
- What happens at creases?

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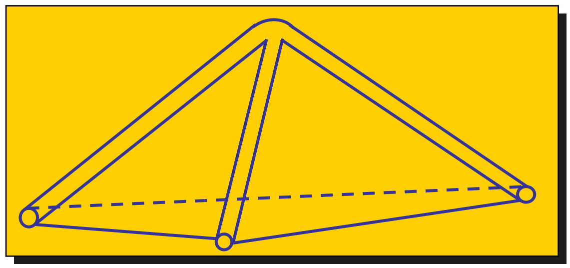
Curvature Estimate: Polyhedral Curvature

- Treat mesh as polyhedron with rounded corners with infinitesimally small radius
- Derive discrete properties from integrals across vertex *region*
  - Convention – associate half of each edge & 1/3 of triangle with vertex



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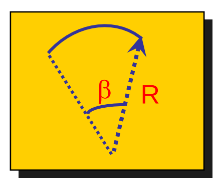
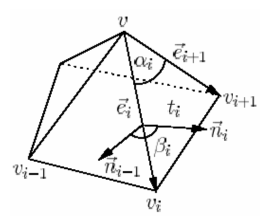
Mean Curvature

- Integral of curvature on circular arc
  - β - central angle

$$\int k = \frac{1}{R} arclength = \frac{1}{R}\frac{\beta}{2\pi} 2\pi R = \beta$$



- On cylindrical parts $H = k_{max}/2$ ($k_{min}=0$)
- On planar faces $H=0$



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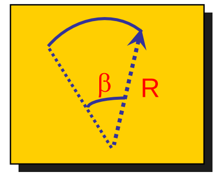
Mean Curvature

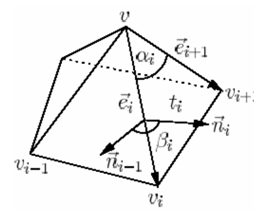
- For entire vertex region

$$\int H = \sum_i \beta_i / 2 \, \| e_i \| / 2 = \frac{1}{4} \sum_i \beta_i \, \| e_i \|$$



- Mean curvature at vertex (A_i triangle area)

$$H = \frac{3}{4 \sum A_i} \sum_i \beta_i \, \| e_i \|$$



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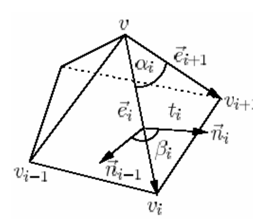
Gaussian Curvature

- Use Gauss-Bonnet Theorem

$$\int_T K = 2\pi - \sum_i \gamma_i - \int_{\partial T} k_{\partial T} = 2\pi - \sum_i \gamma_i$$

- *sum of exterior (jump) angles of polygon around v = sum of face angles at v*
- Curvature at vertex

$$K = \frac{3(2\pi - \sum_i \alpha_i)}{\sum_i A}$$



- Note (Gauss-Bonnet for closed surfaces) – Integral Gaussian curvature ≈ π genus

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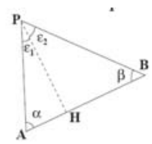
Mean Curvature – Another View

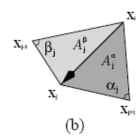
- Mean Curvature Flow $K(x) = kn \approx \frac{\nabla A}{A}$
- Area gradient (per triangle):

$$\nabla A = \frac{1}{2}((cot\ \beta)\ \mathbf{AP} + (cot\ \alpha)\ \mathbf{BP})$$



- Integrate (sum) on all triangles

$$\iint_{\mathcal{A}} \mathbf{K}(\mathbf{x})dA = \frac{1}{2} \sum_{j \in N_1(i)} (cot\ \alpha_{ij} + cot\ \beta_{ij})\ (\mathbf{x}_i - \mathbf{x}_j)$$



- To get K (and normal) divide by “control” area

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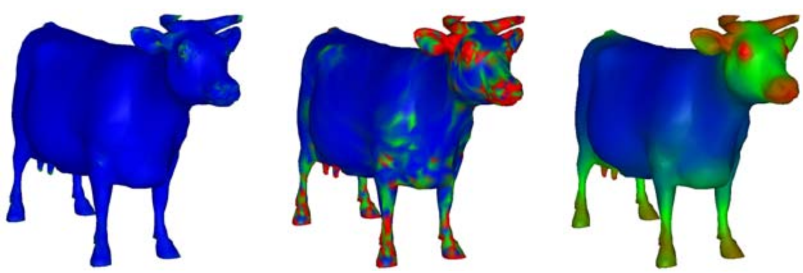
Curvature – Practicalities Example: Gaussian



- Approximation always results in some noise
- Solution
  - Truncate extreme values
    - Can come for instance from division by very small area
  - Smooth
    - More later

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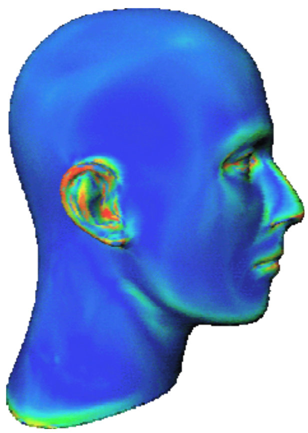
Examples: Mean Curvature



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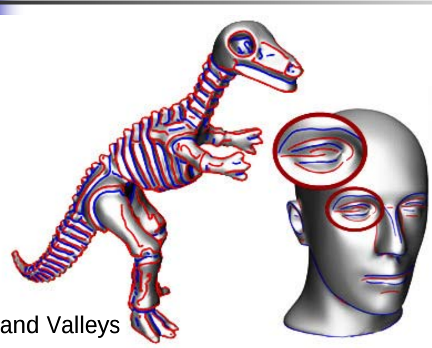
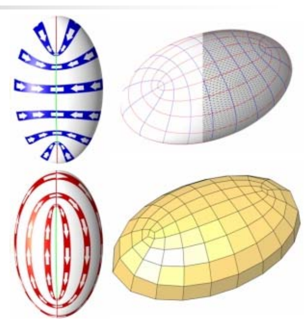
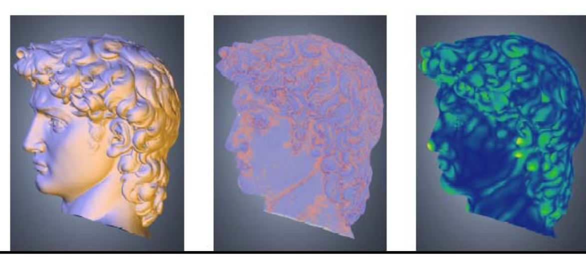
Some Applications



Ridges and Valleys



Remeshing



Viewpoint
selection

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