Differential Geometry & Discrete Operators

Curves

- Tangent vector to curve \( C(t) = (x(t), y(t)) \) is
  \[
  T = C'(t) = \frac{dC(t)}{dt} = [x'(t), y'(t)]
  \]

- Unit length tangent vector
  \[
  \hat{T} = \frac{C'(t)}{|C'(t)|} = \frac{[x'(t), y'(t)]}{\sqrt{x'(t)^2 + y'(t)^2}}
  \]

- Curvature
  \[
  k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}
  \]
Curvature (Curves)

- Curvature is **independent** of parameterization
  - \( C(t), C(t+5), C(2t) \) have same curvature (at corresponding locations)
- Corresponds to radius of osculating circle \( R=1/k \)
- Measure curve bending

Surfaces

- Tangent plane to surface \( S(u,v) \) is spanned by two partials of \( S \):
  - \( \frac{\partial S(u,v)}{\partial u} \)
  - \( \frac{\partial S(u,v)}{\partial v} \)
- **Normal** to surface
  - \( \vec{n} = \frac{\partial S(u,v)}{\partial u} \times \frac{\partial S(u,v)}{\partial v} \)
  - perpendicular to tangent plane
  - Any vector in tangent plane is tangential to \( S(u,v) \)
Curvature

- **Normal curvature** of surface is defined for each tangential direction

- **Principal curvatures** \( K_{min} & K_{max} \): maximum and minimum of normal curvature
  - Correspond to two orthogonal tangent directions
  - Principal directions
  - Not necessarily partial derivative directions
  - Independent of parameterization

3D Curvature

**Isotropic**
Equal in all directions

- \( k_{min} = k_{max} > 0 \)
- \( k_{min} = k_{max} = 0 \)
- \( k_{min} = k_{max} < 0 \)

**Anisotropic**
2 distinct principal directions

- \( k_{min} > 0 \)
- \( k_{max} = 0 \)
- \( k_{min} = 0 \)
- \( k_{max} > 0 \)

Types:
- Spherical
- Planar
- Elliptic
- Parabolic
- Hyperbolic
Curvature

- Typical measures:
  - \textbf{Gaussian} curvature
    \[ K = k_{\min} k_{\max} \]
  - \textbf{Mean} curvature
    \[ H = \frac{k_{\min} + k_{\max}}{2} \]
Curvature on Mesh

- Approximate curvature of (unknown) underlying surface
  - Continuous approximation
    - Approximate the surface & compute continuous differential measures
  - Discrete approximation
    - Approximate differential measures for mesh

Normal Estimation

- Need surface normal to construct approximate surface
  - Defined for each face
  - Solution 1: average face normals
    - Does not reflect face “influence”
  - Solution 2: weighed average of face normals
    - Weights:
      - Face areas
      - Angles at vertex

- What happens at creases?
**Curvature Estimate: Polyhedral Curvature**

- Treat mesh as polyhedron with rounded corners with infinitesimally small radius
- Derive discrete properties from integrals across vertex *region*
  - Convention – associate half of each edge & 1/3 of triangle with vertex

**Mean Curvature**

- Integral of curvature on circular arc
  - $\beta$ - central angle
  - \[ \int k = \frac{1}{R} \text{arclength} = \frac{1}{R} \frac{\beta}{2\pi} \cdot 2\pi R = \beta \]
- On cylindrical parts $H = k_{max}/2$ ($k_{min}=0$)
- On planar faces $H=0$
Mean Curvature

- For entire vertex region
  \[ \int H = \sum_i \beta_i / 2 \| e_i \| / 2 = \frac{1}{4} \sum_i \beta_i \| e_i \| \]

- Mean curvature at vertex \((A_i \text{ triangle area})\)
  \[ H = \frac{3}{4 \sum A_i} \sum \beta_i \| e_i \| \]

Gaussian Curvature

- Use Gauss-Bonnet Theorem
  \[ \int_K = 2\pi - \sum_i \gamma_i - \int_{\partial T} k_{BT} = 2\pi - \sum_i \gamma_i \]

- \( \text{sum of exterior (jump) angles of polygon around } v = \text{sum of face angles at } v \)

- Curvature at vertex
  \[ K = \frac{3(2\pi - \sum_i \alpha_i)}{\sum_i A} \]

- Note (Gauss-Bonnet for closed surfaces) - Integral Gaussian curvature \(\approx \pi \text{ genus} \)
Mean Curvature – Another View

- Mean Curvature Flow  $K(x) = kn \approx \frac{\nabla A}{A}$

- Area gradient (per triangle):
  \[ \nabla A = \frac{1}{2} ((\cot \beta) \ AP + (\cot \alpha) \ BP) \]

- Integrate (sum) on all triangles
  \[ \int_A K(x)dA = \frac{1}{2} \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \ (x_i - x_j) \]

- To get $K$ (and normal) divide by “control” area

Curvature – Practicalities Example: Gaussian

- Approximation always results in some noise
- Solution
  - Truncate extreme values
    - Can come for instance from division by very small area
  - Smooth
    - More later
Examples: Mean Curvature

Some Applications

Ridges and Valleys
Remeshing
Viewpoint selection