SVAN 2016 Mini-Course

Stochastic Convex Optimization Methods in Machine Learning

Mark Schmidt

University of British Columbia, May 2016

www.cs.ubc.ca/~schmidtm/SVAN16

Last Time: Projected-Gradient

- We can convert the non-smooth problem

$$\operatorname*{argmin}_{x \in \mathbb{R}^d} f(x) + \lambda \sum_{g \in G} \|x_g\|_2,$$

into a smooth problem with simple constraints:

$$\operatorname*{argmin}_{x \in \mathbb{R}^d} f(x) + \lambda \sum_{g \in G} r_g, \text{ subject to } r_g \geq \|x_g\|_2 \text{ for all } g.$$

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- With simple constraints, we can use projected-gradient:

$$x^{t+1} = \underset{y \in C}{\operatorname{argmin}} \left\{ f(x^t) + \nabla f(x^t)^T (y - x^t) + \frac{L}{2} \|y - x^t\|^2 \right\},$$

or equivalently projection applied to gradient step:

$$x^{t+1} = \underbrace{\underset{y \in C}{\operatorname{argmin}} \left\{ \|y - x_t^{GD}\| \right\}}_{\text{projection of } x_t^{GD} \text{ onto } \mathcal{C}}, \text{ where } x_t^{GD} = \underbrace{x^t - \alpha_t \nabla f(x^t)}_{\text{gradient step}}.$$

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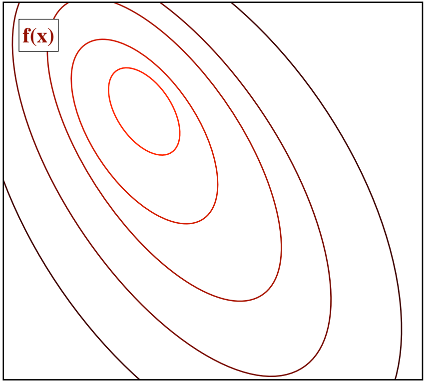

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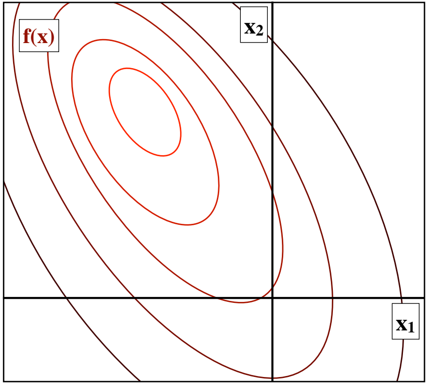

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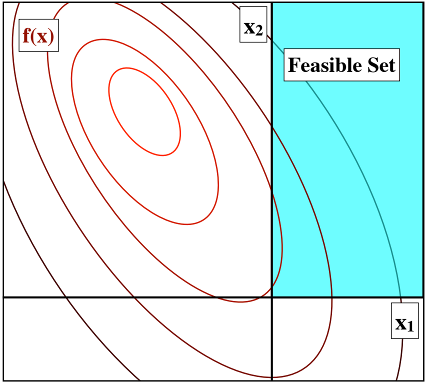

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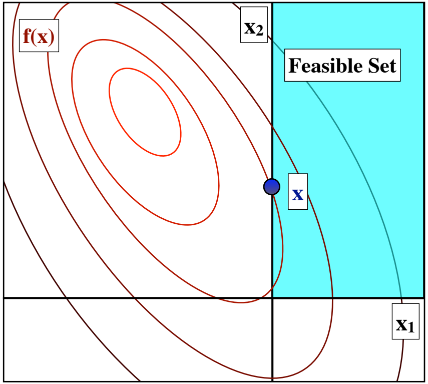

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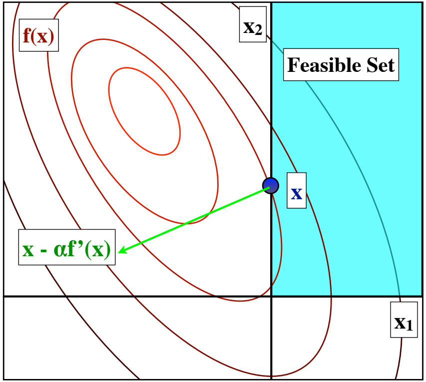

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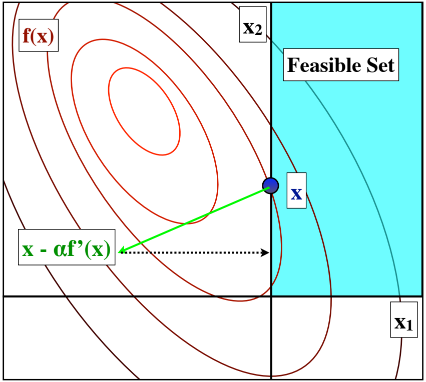

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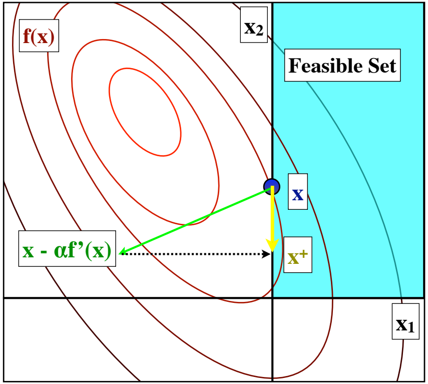

Last Time: Projected-Gradient

- We can convert non-smooth problem into smooth problems with simple constraints:
- But transforming might make problem harder:
  - E.g., transformed problems often lose strong-convexity.
- Can we apply a method like projected-gradient to the original problem?

Gradient Method

- We want to solve a smooth optimization problem:

$$\operatorname*{argmin}_{x \in \mathbb{R}^d} f(x).$$

- Iteration x^t minimizes with quadratic approximation to 'f'':

$$f(y) \approx f(x^t) + \nabla f(x^t)^T(y - x^t) + \frac{L}{2}\|y - x^t\|^2,$$

$$x^{t+1} = \operatorname*{argmin}_{y \in \mathbb{R}^d} \left\{ f(x^t) + \nabla f(x^t)^T(y - x^t) + \frac{L}{2}\|y - x^t\|^2 \right\}.$$

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$$x^{t+1} = \underset{y \in \mathbb{R}^d}{\text{argmin}} \left\{ \frac{1}{2}\|y - (x^t - \alpha_t \nabla f(x^t))\|^2 \right\},$$

and the solution is the gradient algorithm:

$$x^{t+1} = x^t - \alpha_t \nabla f(x^t).$$

Proximal-Gradient Method

- We want to solve a smooth plus non-smooth optimization problem:

$$\operatorname*{argmin}_{x\in\mathbb{R}^d} f(x)+r(x).$$

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- We want to solve a smooth plus non-smooth optimization problem:

$$\underset{x \in \mathbb{R}^d}{\text{argmin}}\, f(x) + r(x).$$

- Iteration x^t minimizes with quadratic approximation to 'f':

$$f(y) + r(y) \approx f(x^t) + \nabla f(x^t)^T(y - x^t) + \frac{L}{2}\|y - x^t\|^2 + r(y),$$

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and the solution is the proximal-gradient algorithm:

$$x^{t+1} = \text{prox}_{\alpha r}[x^t - \alpha_t \nabla f(x^t)].$$

Proximal-Gradient Method

- So proximal-gradient step takes the form:

$$x_t^{GD} = x^t - \alpha_t \nabla f(x^t),$$
$$x^{t+1} = \underset{y \in \mathbb{R}^d}{\text{argmin}} \left\{ \frac{1}{2}\|y - x_t^{GD}\|^2 + \alpha_t r(y) \right\}.$$

- Second part is called the proximal operator with respect to $\alpha_t r$.
- Convergence rates are still the same as for minimizing f alone:
  - E.g, if ∇f is L-Lipschitz, f is μ-strongly convex.and g is convex, then

$$F(x^t) - F(x^*) \leq \left(1 - \frac{\mu}{L}\right)^t \left[F(x^0) - F(x^*)\right],$$

  where $F(x) = f(x) + r(x)$.

Proximal Operator, Iterative Soft Thresholding

- The proximal operator is the solution to

$$\text{prox}_r[x] = \underset{y \in \mathbb{R}^d}{\text{argmin}} \; \frac{1}{2}\|y - x\|^2 + r(y).$$

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  - Apply $x_j =$sign$(x_j) \max\{0, |x_j| - \alpha_t \lambda\}$ element-wise.
  - E.g., if $\alpha_t \lambda = 1$:

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Special case of Projected-Gradient Methods

- Projected-gradient methods are another special case:

$$r(y) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ \infty & \text{if } x \notin \mathcal{C} \end{cases}, \quad (\text{indicator function for convex set } \mathcal{C})$$

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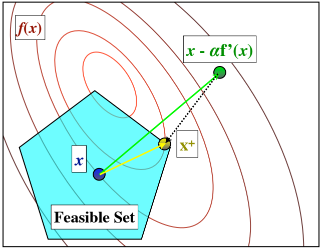
gives

$$x^{t+1} = \mathop{\text{argmin}}_{y \in \mathbb{R}^d} \frac{1}{2}\|y - x\|^2 + r(y) = \mathop{\text{argmin}}_{y \in \mathcal{C}} \frac{1}{2}\|y - x\|^2 = \mathop{\text{argmin}}_{y \in \mathcal{C}} \|y - x\|.$$



Proximal-Gradient for Group L1-Regularization

- The proximal operator for L1-regularization,

$$\operatorname*{argmin}_{y\in\mathbb{R}^d}\left\{\frac{1}{2}\|y-x\|^2+\alpha_t\lambda\|y\|_1\right\},$$

applies soft-threshold element-wise,

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  applies a group soft-threshold group-wise,

$$x_g = \frac{x_g}{\|x_g\|_2} \max\{0, \|x_g\|_2 - \alpha_t \lambda\}.$$

Exact Proximal-Gradient Methods

- We can efficiently compute the proximity operator for:
  1. L1-Regularization and most separable regularizers.
  2. Group ℓ_1-Regularization (disjoint) and most group-separale regularizers.

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  6. Many norm balls and norm cones.
  7. A few other simple regularizers/constraints.

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  7. A few other simple regularizers/constraints.
- Can solve these non-smooth problems as fast as smooth problems.
- But what if we can't efficiently compute proximal operator?

Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximal operator for:
  - Overlapping group L1-regularization.
  - Total-variation regularization.
  - Nuclear-norm regularization.
  - Sums of ‘simple’ functions (proximal-Dykstra).

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  - Nuclear-norm regularization.
  - Sums of 'simple' functions (proximal-Dykstra).
- Inexact proximal-gradient methods:
  - Use an approximation to the proximal operator.
  - If approximation error decreases fast enough, same convergence rate:
    - To get $O(\rho^t)$ rate, error must be in $o(\rho^t)$.

Discussion of Proximal-Gradient

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- With any α_t, proximal–gradient generates a feasible descent direction:
  - If $\bar{x}^t = \text{prox}_{\alpha_t r}[x^t - \alpha_t \nabla f(x^t)]$, then the step

$$x^{t+1} = x^t + \gamma_t(\bar{x}^t - x^t),$$

decreases f and satisfies constraints for γ_t small enough.

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    decreases f and satisfies constraints for γ_t small enough.
- If proximal operator is expensive, can do Armijo line-search for γ_t instead of α_t:
  - Fix α_t and decrease γ_t: "backtracking along the feasible direction".
    - Iterations tend to be in interior.
  - Fix γ_t and decrease α_t: "backtracking along the projection arc".
    - Iterations tend to be at boundary.

Faster Proximal-Gradient Methods

- Accelerated proximal-gradient method:

$$x^{t+1} = \text{prox}_{\alpha_t r}[y^t - \alpha_t \nabla f(x^t)]$$
$$y^{t+1} = x^t + \beta_t(x^{t+1} - x^t).$$

  - Convergence properties same as smooth version.

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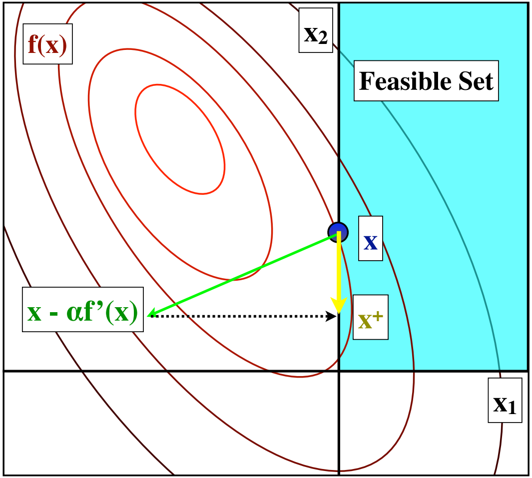
  - Convergence properties same as smooth version.
- The naive Newton-like methods,

$$x^{t+1} = \text{prox}_{\alpha r}[x^t - \alpha_t d^t], \text{ where } d^t \text{ solves } \nabla^2 f(x^t) d^t = \nabla f(x^t),$$

does NOT work.

Naive Projected-Newton



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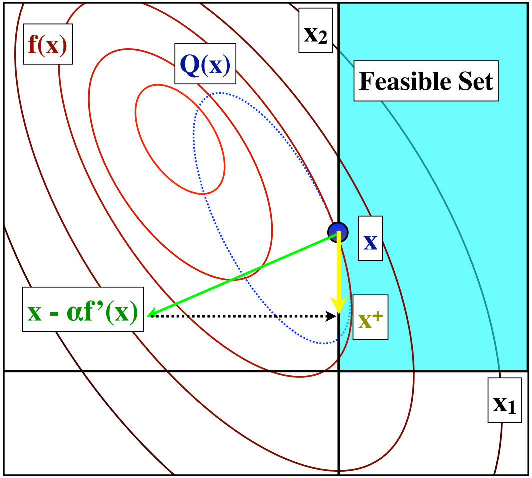

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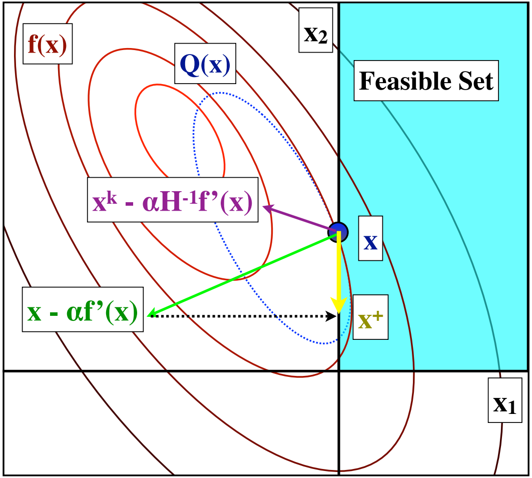

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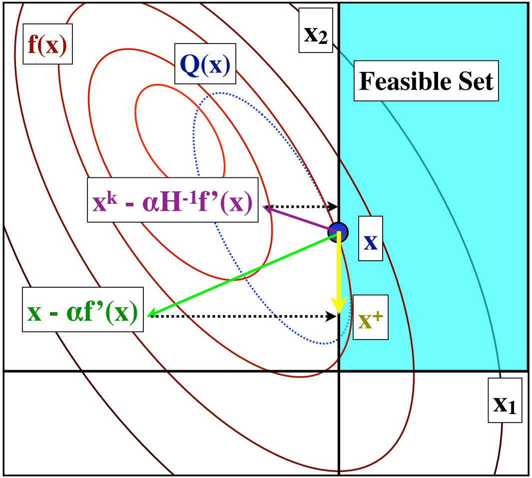

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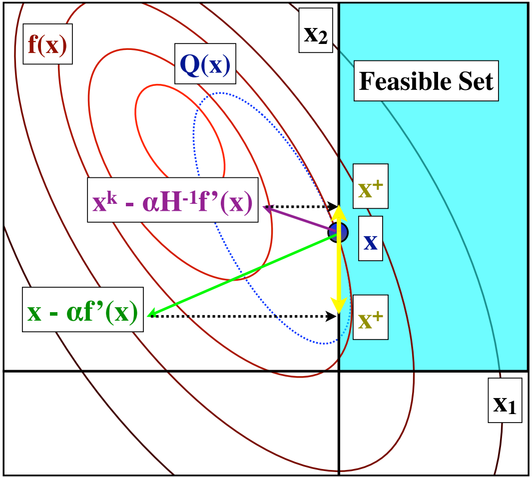

Projected-Newton Method

- Projected-gradient minimizes quadratic approximation,

$$x^{t+1} = \operatorname*{argmin}_{y \in C} \left\{ f(x^t) + \nabla f(x^t)(y - x^t) + \frac{1}{2\alpha_t}\|y - x^t\|^2 \right\}.$$

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- Newton's method can be viewed as quadratic approximation (wth $H^t \approx \nabla^2 f(x^t)$):

$$x^{t+1} = \underset{y \in \mathbb{R}^d}{\operatorname{argmin}} \left\{ f(x^t) + \nabla f(x^t)(y - x^t) + \frac{1}{2\alpha_t}(y - x^t)H^t(y - x^t) \right\}.$$

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- Projected Newton minimizes constrained quadratic approximation:

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Projected-Newton Method

- Projected-gradient minimizes quadratic approximation,

$$x^{t+1} = \operatorname*{argmin}_{y \in C} \left\{ f(x^t) + \nabla f(x^t)(y - x^t) + \frac{1}{2\alpha_t}\|y - x^t\|^2 \right\}.$$

- Newton's method can be viewed as quadratic approximation (wth $H^t \approx \nabla^2 f(x^t)$):

$$x^{t+1} = \operatorname*{argmin}_{y \in \mathbb{R}^d} \left\{ f(x^t) + \nabla f(x^t)(y - x^t) + \frac{1}{2\alpha_t}(y - x^t)H^t(y - x^t) \right\}.$$

- Projected Newton minimizes constrained quadratic approximation:

$$x^{t+1} = \operatorname*{argmin}_{y \in C} \left\{ f(x^t) + \nabla f(x^t)(y - x^t) + \frac{1}{2\alpha_t}(y - x^t)H^t(y - x^t) \right\}.$$

- Equivalently, we project Newton step under different Hessian-defined norm,

$$x^{t+1} = \operatorname*{argmin}_{y \in C} \|y - (x^t - \alpha_t [H^t]^{-1} \nabla f(x^t))\|_{H^t},$$

where general "quadratic norm" is $\|z\|_A = \sqrt{z^T A z}$ for $A \succ 0$.

Discussion of Proximal-Newton

- Proximal-Newton is defined similarly,

$$x^{t+1} = \underset{y \in \mathbb{R}^d}{\text{argmin}} \left\{ f(x^t) + \nabla f(x^t)(y - x^t) + \frac{L}{2}(y - x^t)H^t(y - x^t) + r(y) \right\}.$$

- But this is expensive even when r is simple.

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- But this is expensive even when r is simple.
- There are a variety of practical ways to approximate this:
  - Use Barzilai-Borwein or diagonal H^t.
  - Two-metric projection: special method for separable r.
  - Inexact proximal-Newton: solve the above approximately.
    - Useful when f is very expensive but r is simple.
    - "Costly functions with simple regularizers".

Alternating Direction Method of Multipliers

- Alernating direction method of multipliers (ADMM) solves:

$$\min_{Ax+By=c} f(x) + r(y).$$

- Alternate between prox-like operators with respect to f and r.

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- If prox can not be computed exactly: Linearized ADMM.

Frank-Wolfe Method

- In some cases the projected gradient step

$$x^{t+1} = \underset{y \in \mathcal{C}}{\text{argmin}} \left\{ f(x^t) + \nabla f(x^t)^T (y - x^t) + \frac{1}{2\alpha_t} \|y - x^t\|^2 \right\},$$

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- Frank-Wolfe method simply uses:

$$x^{t+1} = \operatorname*{argmin}_{y \in \mathcal{C}} \left\{ f(x^t) + \nabla f(x^t)^T (y - x^t) \right\},$$

requires compact $\mathcal{C}$, takes convex combination of x^t and x^{t+1}.

- $O(1/t)$ rate for smooth convex objectives, some linear convergence results for strongly-convex [Jaggi, 2013].

Summary

- No black-box method can beat subgradient methods
- For most objectives, you can beat subgradient methods.

Summary

- No black-box method can beat subgradient methods
- For most objectives, you can beat subgradient methods.
- You just need a long list of tricks:
  - Smoothing.
  - Chambolle-Pock.
  - Projected-gradient.
  - Two-metric projection.
  - Proximal-gradient.
  - Proximal-Newton.
  - ADMM
  - Frank-Wolfe.
  - Mirror descent.
  - Incremental surrogate optimization.
  - Solving smooth dual.

Summary

- Group L1-Regularization: encourages sparsity in variable groups.
- Structured sparsity: encourages other patterns in variables.
- Projected-Gradient: allows optimization with simple constraints.
- Proximal-gradient: linear rates for sum of smooth and non-smooth.
- Proximal-Newton: even faster rates in special cases.

- Next time: faster stochastic methods, and kernels for exponential/infinite bases.