Convex Optimization
Machine Learning Summer School

Mark Schmidt

February 2015
Context: Big Data and Big Models

- We are collecting data at unprecedented rates.
  - Seen across many fields of science and engineering.
  - Not gigabytes, but terabytes or petabytes (and beyond).
Context: Big Data and Big Models

- We are collecting data at unprecedented rates.
  - Seen across many fields of science and engineering.
  - Not gigabytes, but terabytes or petabytes (and beyond).

- Many important aspects to the ‘big data’ puzzle:
  - Distributed data storage and management, parallel computation, software paradigms, data mining, machine learning, privacy and security issues, reacting to other agents, power management, summarization and visualization.
Context: Big Data and Big Models

- Machine learning *uses big data to fit richer statistical models*:
  - Vision, bioinformatics, speech, natural language, web, social.
  - Developing broadly applicable tools.
  - Output of models can be used for further analysis.
Context: Big Data and Big Models

- Machine learning uses big data to fit richer statistical models:
  - Vision, bioinformatics, speech, natural language, web, social.
  - Developing broadly applicable tools.
  - Output of models can be used for further analysis.
Context: Big Data and Big Models

- Machine learning uses big data to fit richer statistical models:
  - Vision, bioinformatics, speech, natural language, web, social.
  - Developing broadly applicable tools.
  - Output of models can be used for further analysis.

- Numerical optimization is at the core of many of these models.
Context: Big Data and Big Models

- Machine learning uses big data to fit richer statistical models:
  - Vision, bioinformatics, speech, natural language, web, social.
  - Developing broadly applicable tools.
  - Output of models can be used for further analysis.

- Numerical optimization is at the core of many of these models.
- But, traditional ‘black-box’ methods have difficulty with:
  - the large data sizes.
  - the large model complexities.
Motivation: Why Learn about Convex Optimization?

Why learn about optimization?

- Optimization is at the core of many ML algorithms.
- ML is driving a lot of modern research in optimization.
Motivation: Why Learn about Convex Optimization?

Why learn about optimization?

- Optimization is at the core of many ML algorithms.
- ML is driving a lot of modern research in optimization.

Why in particular learn about convex optimization?

- Among only efficiently-solvable continuous problems.
- You can do a lot with convex models.
  (least squares, lasso, generalized linear models, SVMs, CRFs)
- Empirically effective non-convex methods are often based
  methods with good properties for convex objectives.
  (functions are locally convex around minimizers)
Two Components of My Research

- The first component of my research focuses on computation:
  - We ‘open up the black box’, by using the structure of machine models to derive faster large-scale optimization algorithms.
  - Can lead to enormous speedups for big data and complex models.
Two Components of My Research

- The first component of my research focuses on **computation**:  
  - We ‘open up the black box’, by using the structure of machine models to derive faster large-scale optimization algorithms.  
  - Can lead to enormous speedups for big data and complex models.

- The second component of my research focuses on **modeling**:  
  - By expanding the set of tractable problems, we can propose richer classes of statistical models that can be efficiently fit.
Two Components of My Research

- The first component of my research focuses on computation:
  - We ‘open up the black box’, by using the structure of machine models to derive faster large-scale optimization algorithms.
  - Can lead to enormous speedups for big data and complex models.

- The second component of my research focuses on modeling:
  - By expanding the set of tractable problems, we can propose richer classes of statistical models that can be efficiently fit.

- We can alternate between these two.
Outline

1. Convex Functions
2. Smooth Optimization
3. Non-Smooth Optimization
4. Randomized Algorithms
5. Parallel/Distributed Optimization
Convexity: Zero-order condition

A real-valued function is **convex** if

\[ f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \]

for all \( x, y \in \mathbb{R}^n \) and all \( 0 \leq \theta \leq 1 \).

- Function is *below a linear interpolation* from \( x \) to \( y \).
- Implies that all local minima are global minima.
  
  (contradiction otherwise)
Convexity: Zero-order condition

A real-valued function is convex if

\[ f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \]

for all \( x, y \in \mathbb{R}^n \) and all \( 0 \leq \theta \leq 1 \).

- Function is below a linear interpolation from \( x \) to \( y \).
- Implies that all local minima are global minima.

(contradiction otherwise)
Convexity: Zero-order condition

A real-valued function is **convex** if

\[
f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y),
\]

for all \( x, y \in \mathbb{R}^n \) and all \( 0 \leq \theta \leq 1 \).

- Function is **below a linear interpolation** from \( x \) to \( y \).
- Implies that all local minima are global minima.

(contradiction otherwise)
Convexity: Zero-order condition

A real-valued function is **convex** if

\[ f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \]

for all \( x, y \in \mathbb{R}^n \) and all \( 0 \leq \theta \leq 1 \).

- Function is *below a linear interpolation* from \( x \) to \( y \).
- Implies that all local minima are global minima.

(contradiction otherwise)
A real-valued function is **convex** if

\[
f(\theta x + (1 \theta)y) \leq \theta f(x) + (1 - \theta)f(y),
\]

for all \(x, y \in \mathbb{R}^n\) and all \(0 \leq \theta \leq 1\).

- Function is **below a linear interpolation** from \(x\) to \(y\).
- Implies that all local minima are global minima.

(contradiction otherwise)
A real-valued function is **convex** if

\[ f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \]

for all \( x, y \in \mathbb{R}^n \) and all \( 0 \leq \theta \leq 1 \).

- Function is **below a linear interpolation** from \( x \) to \( y \).
- Implies that all local minima are global minima.

(contradiction otherwise)
Convexity: Zero-order condition

A real-valued function is \textit{convex} if

\[ f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \]

for all \( x, y \in \mathbb{R}^n \) and all \( 0 \leq \theta \leq 1 \).

- Function is \textit{below a linear interpolation} from \( x \) to \( y \).
- Implies that all local minima are global minima.

(contradiction otherwise)
Convexity: Zero-order condition

A real-valued function is **convex** if

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y),$$

for all $x, y \in \mathbb{R}^n$ and all $0 \leq \theta \leq 1$.

- Function is *below a linear interpolation* from $x$ to $y$.
- Implies that all local minima are global minima.

(contradiction otherwise)
We say that a function $f$ is a norm if:

1. $f(0) = 0$.
2. $f(\theta x) = |\theta| f(x)$.
3. $f(x + y) \leq f(x) + f(y)$.

Examples:

$$\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x^T x}$$

$$\|x\|_1 = \sum_i |x_i|$$

$$\|x\|_H = \sqrt{x^T H x}$$
Convexity of Norms

We say that a function $f$ is a norm if:

1. $f(0) = 0$.
2. $f(\theta x) = |\theta| f(x)$.
3. $f(x + y) \leq f(x) + f(y)$.

Examples:

\[
\|x\|_2 = \sqrt{\sum_i x_i^2} = \sqrt{x^T x}
\]

\[
\|x\|_1 = \sum_i |x_i|
\]

\[
\|x\|_H = \sqrt{x^T H x}
\]

Norms are convex:

\[
f(\theta x + (1 - \theta)y) \leq f(\theta x) + f((1 - \theta)y) \quad (3)
\]

\[
= \theta f(x) + (1 - \theta) f(y) \quad (2)
\]
Strict Convexity

A real-valued function is strictly convex if

\[ f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y), \]

for all \( x \neq y \in \mathbb{R}^n \) and all \( 0 < \theta < 1 \).

- Strictly below the linear interpolation from \( x \) to \( y \).
Strict Convexity

A real-valued function is **strictly convex** if

\[ f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y), \]

for all \( x \neq y \in \mathbb{R}^n \) and all \( 0 < \theta < 1 \).

- **Strictly below the linear interpolation** from \( x \) to \( y \).
- Implies at most one global minimum.

(otherwise, could construct lower global minimum)
A real-valued differentiable function is convex iff

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \]

for all \( x, y \in \mathbb{R}^n \).

- The function is globally above the tangent at \( x \).
  
  (if \( \nabla f(y) = 0 \) then \( y \) is a global minimizer)
A real-valued differentiable function is convex iff

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \]

for all \( x, y \in \mathbb{R}^n. \)

- The function is globally above the tangent at \( x. \)
  (if \( \nabla f(y) = 0 \) then \( y \) is a a global minimizer)
Convexity: First-order condition

A real-valued differentiable function is convex iff

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \]

for all \( x, y \in \mathbb{R}^n \).

- The function is globally above the tangent at \( x \).

  (if \( \nabla f(y) = 0 \) then \( y \) is a global minimizer)
Convexity: First-order condition

A real-valued differentiable function is convex iff

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \]

for all \( x, y \in \mathbb{R}^n \).

- The function is globally above the tangent at \( x \).
  
  (if \( \nabla f(y) = 0 \) then \( y \) is a a global minimizer)
Convex Functions  Smooth Optimization  Non-Smooth Optimization  Randomized Algorithms  Parallel/Distributed Optimization

Convexity: Second-order condition

A real-valued twice-differentiable function is convex iff

\[ \nabla^2 f(x) \succeq 0 \]

for all \( x \in \mathbb{R}^n \).

- The function is flat or curved upwards in every direction.
Convexity: Second-order condition

A real-valued twice-differentiable function is convex iff

$$\nabla^2 f(x) \succeq 0$$

for all $x \in \mathbb{R}^n$.

- The function is flat or curved upwards in every direction.

A real-valued function $f$ is a quadratic if it can be written in the form:

$$f(x) = \frac{1}{2} x^T A x + b^T x + c.$$ 

Since $\nabla^2 f(x) = A$, it is convex if $A \succeq 0$. 
E.g., least squares has $\nabla^2 f(x) = A^T A \succeq 0$. 
Examples of Convex Functions

Some simple convex functions:

- $f(x) = c$
- $f(x) = a^T x$
- $f(x) = ax^2 + b$ (for $a > 0$)
- $f(x) = \exp(ax)$
- $f(x) = x \log x$ (for $x > 0$)
- $f(x) = \|x\|^2$
- $f(x) = \max_i \{x_i\}$
Examples of Convex Functions

Some simple convex functions:

- $f(x) = c$
- $f(x) = a^T x$
- $f(x) = ax^2 + b$ (for $a > 0$)
- $f(x) = \exp(ax)$
- $f(x) = x \log x$ (for $x > 0$)
- $f(x) = \|x\|^2$
- $f(x) = \max_i \{x_i\}$

Some other notable examples:

- $f(x, y) = \log(e^x + e^y)$
- $f(X) = \log \det X$ (for $X$ positive-definite).
- $f(x, Y) = x^T Y^{-1} x$ (for $Y$ positive-definite)
Operations that Preserve Convexity

1. Non-negative weighted sum:

\[ f(x) = \theta_1 f_1(x) + \theta_2 f_2(x). \]

2. Composition with affine mapping:

\[ g(x) = f(Ax + b). \]

3. Pointwise maximum:

\[ f(x) = \max_i \{f_i(x)\}. \]
Operations that Preserve Convexity

1. Non-negative weighted sum:

\[ f(x) = \theta_1 f_1(x) + \theta_2 f_2(x). \]

2. Composition with affine mapping:

\[ g(x) = f(Ax + b). \]

3. Pointwise maximum:

\[ f(x) = \max_i \{f_i(x)\}. \]

Show that least-residual problems are convex for any \( \ell_p \)-norm:

\[ f(x) = \|Ax - b\|_p \]
Operations that Preserve Convexity

1. Non-negative weighted sum:

\[ f(x) = \theta_1 f_1(x) + \theta_2 f_2(x). \]

2. Composition with affine mapping:

\[ g(x) = f(Ax + b). \]

3. Pointwise maximum:

\[ f(x) = \max_i \{ f_i(x) \}. \]

Show that least-residual problems are convex for any \( \ell_p \)-norm:

\[ f(x) = \| Ax - b \|_p \]

We know that \( \| \cdot \|_p \) is a norm, so it follows from (2).
Operations that Preserve Convexity

1. Non-negative weighted sum:
   \[ f(x) = \theta_1 f_1(x) + \theta_2 f_2(x). \]

2. Composition with affine mapping:
   \[ g(x) = f(Ax + b). \]

3. Pointwise maximum:
   \[ f(x) = \max_i \{ f_i(x) \}. \]

Show that SVMs are convex:

\[ f(x) = \frac{1}{2} \|x\|^2 + C \sum_{i=1}^{n} \max\{0, 1 - b_i a_i^T x\}. \]
Operations that Preserve Convexity

1. Non-negative weighted sum:

\[ f(x) = \theta_1 f_1(x) + \theta_2 f_2(x). \]

2. Composition with affine mapping:

\[ g(x) = f(Ax + b). \]

3. Pointwise maximum:

\[ f(x) = \max_i \{ f_i(x) \}. \]

Show that SVMs are convex:

\[ f(x) = \frac{1}{2} \|x\|^2 + C \sum_{i=1}^n \max \{ 0, 1 - b_i a_i^T x \}. \]

The first term has Hessian \( I > 0 \), for the second term use (3) on the two (convex) arguments, then use (1) to put it all together.
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is **Lipschitz-continuous**: (can not change too quickly)

  $$|f(x) - f(y)| \leq L\|x - y\|.$$
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is Lipschitz-continuous: (can not change too quickly)

  $$|f(x) - f(y)| \leq L\|x - y\|.$$
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is **Lipschitz-continuous**: (can not change too quickly)

$$|f(x) - f(y)| \leq L\|x - y\|.$$

How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is **Lipschitz-continuous**: (can not change too quickly)

  $$|f(x) - f(y)| \leq L\|x - y\|.$$
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is **Lipschitz-continuous**: (can not change too quickly)

  $$|f(x) - f(y)| \leq L\|x - y\|.$$
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is **Lipschitz-continuous**: (can not change too quickly)
  
  $$|f(x) - f(y)| \leq L\|x - y\|.$$ 

- After $t$ iterations, the error of *any algorithm* is $\Omega(1/t^{1/n})$.
  
  (this is in the worst case, and note that grid-search is nearly optimal)
How hard is real-valued optimization?

How long to find an $\epsilon$-optimal minimizer of a real-valued function?

$$\min_{x \in \mathbb{R}^n} f(x).$$

- General function: impossible!
  
  (think about arbitrarily small value at some infinite decimal expansion)

We need to make some assumptions about the function:

- Assume $f$ is **Lipschitz-continuous**: (can not change too quickly)
  $$|f(x) - f(y)| \leq L\|x - y\|.$$

- After $t$ iterations, the error of any algorithm is $\Omega(1/t^{1/n})$.
  
  (this is in the worst case, and note that grid-search is nearly optimal)

- Optimization is hard, but assumptions make a big difference.
  
  (we went from impossible to very slow)
Motivation for First-Order Methods

- Well-known that we can solve convex optimization problems in polynomial-time by interior-point methods.
Motivation for First-Order Methods

- Well-known that we can solve \textit{convex} optimization problems in polynomial-time by interior-point methods.
- However, these solvers require $O(n^2)$ or worse cost per iteration.
  - Infeasible for applications where $n$ may be in the billions.
Motivation for First-Order Methods

- Well-known that we can solve convex optimization problems in polynomial-time by interior-point methods.
- However, these solvers require $O(n^2)$ or worse cost per iteration.
  - Infeasible for applications where $n$ may be in the billions.
- Solving big problems has led to re-newed interest in simple first-order methods (gradient methods):
  \[ x^+ = x - \alpha \nabla f(x). \]

- These only have $O(n)$ iteration costs.
- But we must analyze how many iterations are needed.
\textbf{$\ell_2$-Regularized Logistic Regression}

- Consider $\ell_2$-regularized logistic regression:

$$f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))) + \frac{\lambda}{2} \|x\|^2.$$  

- Objective $f$ is convex.
- First term is Lipschitz continuous.
- Second term is not Lipschitz continuous.
Consider $\ell_2$-regularized logistic regression:

$$f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))) + \frac{\lambda}{2} \|x\|^2.$$

- Objective $f$ is convex.
- First term is Lipschitz continuous.
- Second term is not Lipschitz continuous.
- But we have

$$\mu I \leq \nabla^2 f(x) \leq L I.$$

$$(L = \frac{1}{4} \|A\|_2^2 + \lambda, \mu = \lambda)$$

- Gradient is Lipschitz-continuous.
- Function is strongly-convex.

(implies strict convexity, and existence of unique solution)
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some $z$ we have:

$$f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some $z$ we have:

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x)$$

- Use that $\nabla^2 f(z) \preceq LI$.

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2$$

- Global quadratic upper bound on function value.
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some $z$ we have:
  \[ f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x) \]

- Use that $\nabla^2 f(z) \preceq L I$.

  \[ f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2 \]

- Global quadratic upper bound on function value.
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some \( z \) we have:

\[
f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x)
\]

- Use that \( \nabla^2 f(z) \preceq L I \).

\[
f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2
\]

- Global quadratic upper bound on function value.
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some \( z \) we have:
  \[
f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)
  \]
- Use that \( \nabla^2 f(z) \preceq L I \).
  \[
f(y) \leq f(x) + \nabla f(x)^T(y - x) + \frac{L}{2} \|y - x\|^2
  \]
- **Global quadratic upper bound on function value.**
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some $z$ we have:

\[
f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x)
\]

- Use that $\nabla^2 f(z) \preceq LI$.

\[
f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2
\]

- Global quadratic upper bound on function value.
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some $z$ we have:
  \[
  f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)
  \]
  - Use that $\nabla^2 f(z) \preceq LI$.
  
  \[
  f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2
  \]
  - Global quadratic upper bound on function value.
Properties of Lipschitz-Continuous Gradient

- From Taylor’s theorem, for some \( z \) we have:
  \[
f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)
\]

- Use that \( \nabla^2 f(z) \preceq LL \).
  \[
f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2} \|y - x\|^2
\]

- Global quadratic upper bound on function value.
- Set \( x^+ \) to minimize upper bound in terms of \( y \):
  \[
x^+ = x - \frac{1}{L} \nabla f(x).
\]
  (gradient descent with step-size of \( 1/L \))

- Plugging this value in:
  \[
f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2.
\]
  (decrease of at least \( \frac{1}{2L} \|\nabla f(x)\|^2 \))
Properties of Strong-Convexity

- From Taylor’s theorem, for some $z$ we have:

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$
Properties of Strong-Convexity

- From Taylor’s theorem, for some $z$ we have:
  \[ f(y) = f(x) + \nabla f(x)^T(y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x) \]

- Use that $\nabla^2 f(z) \succeq \mu I$.
  \[ f(y) \geq f(x) + \nabla f(x)^T(y - x) + \frac{\mu}{2}\|y - x\|^2 \]

- **Global quadratic lower bound on function value.**
Properties of Strong-Convexity

- From Taylor’s theorem, for some $z$ we have:
  
  $$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x)$$

- Use that $\nabla^2 f(z) \succeq \mu I$.

  $$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|^2$$

- Global quadratic lower bound on function value.
Properties of Strong-Convexity

- From Taylor’s theorem, for some z we have:
  \[ f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x) \]

- Use that \( \nabla^2 f(z) \succeq \mu I \).
  \[ f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \| y - x \|^2 \]

- Global quadratic lower bound on function value.
Properties of Strong-Convexity

- From Taylor’s theorem, for some $z$ we have:
  \[ f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z)(y - x) \]

- Use that $\nabla^2 f(z) \succeq \mu I$.
  \[ f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|^2 \]

- Global quadratic lower bound on function value.
Properties of Strong-Convexity

- From Taylor’s theorem, for some $z$ we have:

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(z) (y - x)$$

- Use that $\nabla^2 f(z) \succeq \mu I$.

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\mu}{2} \|y - x\|^2$$

- Global quadratic lower bound on function value.
- Minimize both sides in terms of $y$:

$$f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$

- Upper bound on how far we are from the solution.
Linear Convergence of Gradient Descent

- We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$
Linear Convergence of Gradient Descent

We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$
We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$
Linear Convergence of Gradient Descent

We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \| \nabla f(x) \|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu} \| \nabla f(x) \|^2.$$
Linear Convergence of Gradient Descent

We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$
We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2,$$

$$f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$
We have bounds on $x^+$ and $x^*$:

\[
f(x^+) \leq f(x) - \frac{1}{2L} \| \nabla f(x) \|^2, \quad f(x^*) \geq f(x) - \frac{1}{2 \mu} \| \nabla f(x) \|^2.
\]

Combine them to get

\[
f(x^+) - f(x^*) \leq \left( 1 - \frac{\mu}{L} \right) [f(x) - f(x^*)]
\]
Linear Convergence of Gradient Descent

- We have bounds on $x^+$ and $x^*$:

$$f(x^+) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2, \quad f(x^*) \geq f(x) - \frac{1}{2\mu} \|\nabla f(x)\|^2.$$

Combine them to get

$$f(x^+) - f(x^*) \leq \left(1 - \frac{\mu}{L}\right) [f(x) - f(x^*)]$$

- This gives a linear convergence rate:

$$f(x^t) - f(x^*) \leq \left(1 - \frac{\mu}{L}\right)^t [f(x^0) - f(x^*)]$$

- Each iteration multiplies the error by a fixed amount. (very fast if $\mu/L$ is not too close to one)
Maximum Likelihood Logistic Regression

What about maximum-likelihood logistic regression?

\[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]
Maximum Likelihood Logistic Regression

- What about maximum-likelihood logistic regression?

\[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]

- We now only have

\[ 0 \leq \nabla^2 f(x) \leq L I. \]

- **Convexity** only gives a linear upper bound on \( f(x^*) \):

\[ f(x^*) \leq f(x) + \nabla f(x)^T (x^* - x) \]
**Maximum Likelihood Logistic Regression**

- What about *maximum-likelihood logistic regression*?

  \[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]

- We now only have

  \[ 0 \leq \nabla^2 f(x) \leq L I. \]

- **Convexity** only gives a linear upper bound on \( f(x^*) \):

  \[ f(x^*) \leq f(x) + \nabla f(x)^T(x^* - x) \]
Maximum Likelihood Logistic Regression

- What about maximum-likelihood logistic regression?

\[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]

- We now only have

\[ 0 \leq \nabla^2 f(x) \leq L I. \]

- Convexity only gives a linear upper bound on \( f(x^*) \):

\[ f(x^*) \leq f(x) + \nabla f(x)^T (x^* - x) \]
Maximum Likelihood Logistic Regression

What about maximum-likelihood logistic regression?

\[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]

We now only have

\[ 0 \leq \nabla^2 f(x) \leq L I. \]

Convexity only gives a linear upper bound on \( f(x^*) \):

\[ f(x^*) \leq f(x) + \nabla f(x)^T (x^* - x) \]
Maximum Likelihood Logistic Regression

- Consider maximum-likelihood logistic regression:
  \[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]
- We now only have
  \[ 0 \preceq \nabla^2 f(x) \preceq L I. \]
- Convexity only gives a linear upper bound on \( f(x^*) \):
  \[ f(x^*) \leq f(x) + \nabla f(x)^T (x^* - x) \]
- If some \( x^* \) exists, we have the sublinear convergence rate:
  \[ f(x^t) - f(x^*) = O(1/t) \]
  (compare to slower \( \Omega(1/t^{-1/N}) \) for general Lipschitz functions)
Consider maximum-likelihood logistic regression:

\[ f(x) = \sum_{i=1}^{n} \log(1 + \exp(-b_i(x^T a_i))). \]

We now only have

\[ 0 \leq \nabla^2 f(x) \leq L I. \]

Convexity only gives a linear upper bound on \( f(x^*) \):

\[ f(x^*) \leq f(x) + \nabla f(x)^T(x^* - x) \]

If some \( x^* \) exists, we have the sublinear convergence rate:

\[ f(x^t) - f(x^*) = O(1/t) \]

(compare to slower \( \Omega(1/t^{-1/N}) \) for general Lipschitz functions)

If \( f \) is convex, then \( f + \lambda \|x\|^2 \) is strongly-convex.
Gradient Method: Practical Issues

- In practice, searching for step size (line-search) is usually much faster than $\alpha = 1/L$.

  (and doesn’t require knowledge of $L$)
Gradient Method: Practical Issues

- In practice, searching for step size (line-search) is usually much faster than $\alpha = 1/L$.
  (and doesn’t require knowledge of $L$)

- Basic Armijo backtracking line-search:
  1. Start with a large value of $\alpha$.
  2. Divide $\alpha$ in half until we satisfy (typically value is $\gamma = .0001$)

\[
f(x^+) \leq f(x) - \gamma \alpha \|\nabla f(x)\|^2.
\]
Gradient Method: Practical Issues

• In practice, searching for step size (line-search) is usually much faster than $\alpha = 1/L$. (and doesn’t require knowledge of $L$)

• Basic Armijo backtracking line-search:
  1. Start with a large value of $\alpha$.
  2. Divide $\alpha$ in half until we satisfy (typically value is $\gamma = .0001$)

\[
f(x^+) \leq f(x) - \gamma \alpha \|\nabla f(x)\|^2.
\]

• Practical methods may use Wolfe conditions (so $\alpha$ isn’t too small), and/or use interpolation to propose trial step sizes.
  (with good interpolation, $\approx 1$ evaluation of $f$ per iteration)
Gradient Method: Practical Issues

- In practice, searching for step size (line-search) is usually much faster than $\alpha = 1/L$. (and doesn’t require knowledge of $L$)

- Basic Armijo backtracking line-search:
  1. Start with a large value of $\alpha$.
  2. Divide $\alpha$ in half until we satisfy (typically value is $\gamma = .0001$)
     \[
     f(x^+) \leq f(x) - \gamma \alpha \|\nabla f(x)\|^2.
     \]

- Practical methods may use Wolfe conditions (so $\alpha$ isn’t too small), and/or use interpolation to propose trial step sizes.
  (with good interpolation, $\approx 1$ evaluation of $f$ per iteration)

- Also, check your derivative code!

\[
\nabla_i f(x) \approx \frac{f(x + \delta e_i) - f(x)}{\delta}
\]

- For large-scale problems you can check a random direction $d$:

\[
\nabla f(x)^T d \approx \frac{f(x + \delta d) - f(x)}{\delta}
\]
We are going to explore the ‘convex optimization zoo’:

- Gradient method for smooth/convex: $O(1/t)$.
- Gradient method for smooth/strongly-convex: $O((1 - \mu/L)^t)$. 

Error bounds exist that give linear convergence without strong-convexity. [Luo & Tseng, 1993]. Is this the best algorithm under these assumptions?
We are going to explore the ‘convex optimization zoo’:

- Gradient method for smooth/convex: $O(1/t)$.
- Gradient method for smooth/strongly-convex: $O((1 − \mu/L)^t)$.

Rates are the same if only once-differentiable.

Line-search doesn’t change the worst-case rate.

(Strongly-convex slightly improved with $\alpha = 2/(\mu + L)$)
We are going to explore the ‘convex optimization zoo’:

- Gradient method for smooth/convex: $O(1/t)$.
- Gradient method for smooth/strongly-convex: $O((1 - \mu/L)^t)$.

Rates are the same if only once-differentiable.

Line-search doesn’t change the worst-case rate.

(strongly-convex slightly improved with $\alpha = 2/(\mu + L)$)

‘Error bounds’ exist that give linear convergence without strong-convexity. [Luo & Tseng, 1993].
We are going to explore the ‘convex optimization zoo’:

- Gradient method for smooth/convex: $O(1/t)$.
- Gradient method for smooth/strongly-convex: $O((1 - \mu/L)^t)$.

Rates are the same if only once-differentiable.

- Line-search doesn’t change the worst-case rate.
  
  (strongly-convex slightly improved with $\alpha = 2/(\mu + L)$)

- ‘Error bounds’ exist that give linear convergence without strong-convexity. [Luo & Tseng, 1993].

- Is this the best algorithm under these assumptions?
Accelerated Gradient Method

- Nesterov’s accelerated gradient method:

  \[ x_{t+1} = y_t - \alpha_t \nabla f(y_t), \]
  \[ y_{t+1} = x_t + \beta_t (x_{t+1} - x_t), \]

  for appropriate \( \alpha_t, \beta_t \).
Accelerated Gradient Method

- Nesterov’s accelerated gradient method:

\[ x_{t+1} = y_t - \alpha_t \nabla f(y_t), \]
\[ y_{t+1} = x_t + \beta_t (x_{t+1} - x_t), \]

for appropriate \( \alpha_t, \beta_t \).

- Motivation: “to make the math work”

  (but similar to heavy-ball/momentum and conjugate gradient method)
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Strongly-Convex</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Strongly-Convex</td>
<td>$O((1 - \sqrt{\mu/L})^t)$</td>
</tr>
</tbody>
</table>

- $O(1/t^2)$ is optimal given only these assumptions. (sometimes called the *optimal* gradient method)
- The faster linear convergence rate is close to optimal.
- Also faster in practice, but implementation details matter.
Newton’s Method

- The oldest differentiable optimization method is **Newton’s**. (also called IRLS for functions of the form $f(Ax)$)

- Modern form uses the update

  $$x^+ = x - \alpha d,$$

  where $d$ is a solution to the system

  $$\nabla^2 f(x) d = \nabla f(x).$$  \hspace{1cm} (Assumes $\nabla^2 f(x) \succ 0$)

We can generalize the Armijo condition to

$$f(x^+) \leq f(x) + \gamma \alpha \nabla f(x)^T d.$$  

Has a natural step length of $\alpha = 1$. (always accepted when close to a minimizer)
Newton’s Method

- The oldest differentiable optimization method is **Newton’s**. (also called IRLS for functions of the form \( f(Ax) \))
- Modern form uses the update
  \[
  x^+ = x - \alpha d,
  \]
  where \( d \) is a solution to the system
  \[
  \nabla^2 f(x) d = \nabla f(x). \quad \text{(Assumes } \nabla^2 f(x) \succ 0)\]
- Equivalent to minimizing the quadratic approximation:
  \[
  f(y) \approx f(x) + \nabla f(x)^T(y - x) + \frac{1}{2\alpha} \| y - x \|^2_{\nabla^2 f(x)}. 
  \]
  (recall that \( \| x \|_H^2 = x^T H x \))
Newton’s Method

- The oldest differentiable optimization method is **Newton’s**.
  (also called IRLS for functions of the form $f(Ax)$)
- Modern form uses the update
  $$x^+ = x - \alpha d,$$
  where $d$ is a solution to the system
  $$\nabla^2 f(x) d = \nabla f(x).$$
  (Assumes $\nabla^2 f(x) \succ 0$)
- Equivalent to minimizing the quadratic approximation:
  $$f(y) \approx f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2_{\nabla^2 f(x)}.$$
  (recall that $\|x\|^2_H = x^T H x$)
- We can generalize the Armijo condition to
  $$f(x^+) \leq f(x) + \gamma \alpha \nabla f(x)^T d.$$
  (always accepted when close to a minimizer)
Newton’s Method
Newton’s Method

\[ f(x) + \nabla f(x)^T (y-x) + \frac{1}{2} (y-x)^T \nabla^2 f(x) (y-x) \]
Newton’s Method

\[ f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T\nabla^2 f(x)(y-x) \]
Newton’s Method
Newton’s Method
Newton’s Method

\[ f(x) \]

\[ x^k - \alpha \nabla f(x^k) \]
Newton’s Method

The diagram illustrates Newton’s Method for optimization. The function $f(x)$ is shown, along with the point $x^k$ and the quadratic approximation $Q(x, \alpha)$ centered at $x^k$. The update step is $x^{k+1} = x^k - \alpha \nabla f(x^k)$, moving towards a minimum of the function.
Newton’s Method
Convergence Rate of Newton’s Method

- If $\nabla^2 f(x)$ is Lipschitz-continuous and $\nabla^2 f(x) \succeq \mu$, then close to $x^*$ Newton’s method has local superlinear convergence:

$$f(x^{t+1}) - f(x^*) \leq \rho_t [f(x^t) - f(x^*)],$$

with $\lim_{t \to \infty} \rho_t = 0$.

- Converges very fast, use it if you can!

- But requires solving $\nabla^2 f(x)d = \nabla f(x)$. 
Convergence Rate of Newton’s Method

- If $\nabla^2 f(x)$ is Lipschitz-continuous and $\nabla^2 f(x) \succeq \mu$, then close to $x^*$ Newton’s method has local superlinear convergence:

$$f(x^{t+1}) - f(x^*) \leq \rho_t [f(x^t) - f(x^*)],$$

with $\lim_{t \to \infty} \rho_t = 0$.

- Converges very fast, use it if you can!

- But requires solving $\nabla^2 f(x) d = \nabla f(x)$.

- Get global rates under various assumptions (cubic-regularization/accelerated/self-concordant).
Newton’s Method: Practical Issues

There are many practical variants of Newton’s method:

- Modify the Hessian to be positive-definite.
- Only compute the Hessian every $m$ iterations.
- Only use the diagonals of the Hessian.
- **Quasi-Newton:** Update a (diagonal plus low-rank) approximation of the Hessian (BFGS, L-BFGS).
Newton’s Method: Practical Issues

There are many practical variants of Newton’s method:

- Modify the Hessian to be positive-definite.
- Only compute the Hessian every \( m \) iterations.
- Only use the diagonals of the Hessian.
- **Quasi-Newton**: Update a (diagonal plus low-rank) approximation of the Hessian (BFGS, L-BFGS).
- **Hessian-free**: Compute \( d \) inexactlly using Hessian-vector products:
  \[
  \nabla^2 f(x)d = \lim_{\delta \to 0} \frac{\nabla f(x + \delta d) - \nabla f(x)}{\delta}
  \]

- **Barzilai-Borwein**: Choose a step-size that acts like the Hessian over the last iteration:
  \[
  \alpha = \frac{(x^+ - x)^T(\nabla f(x^+) - \nabla f(x))}{\|\nabla f(x^+) - f(x)\|^2}
  \]

Another related method is **nonlinear conjugate gradient**.
Outline

1. Convex Functions
2. Smooth Optimization
3. Non-Smooth Optimization
4. Randomized Algorithms
5. Parallel/Distributed Optimization
Motivation: Sparse Regularization

- Consider $\ell_1$-regularized optimization problems,

$$\min_{x} f(x) + \lambda \|x\|_1,$$

where $f$ is differentiable.

- For example, $\ell_1$-regularized least squares,

$$\min_{x} \|Ax - b\|^2 + \lambda \|x\|_1$$

- Regularizes and encourages sparsity in $x$
Motivation: Sparse Regularization

- Consider $\ell_1$-regularized optimization problems,
  \[
  \min_x f(x) + \lambda \|x\|_1,
  \]
  where $f$ is differentiable.
- For example, $\ell_1$-regularized least squares,
  \[
  \min_x \|Ax - b\|^2 + \lambda \|x\|_1
  \]
  Regularizes and encourages sparsity in $x$
  The objective is non-differentiable when any $x_i = 0$.
  How can we solve non-smooth convex optimization problems?
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

A vector \( d \) is a *subgradient* of a convex function \( f \) at \( x \) if

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for \textit{differentiable} convex functions we have

$$f(y) \geq f(x) + \nabla f(x)^T(y - x), \forall x, y.$$  

A vector \(d\) is a \textit{subgradient} of a convex function \(f\) at \(x\) if

$$f(y) \geq f(x) + d^T(y - x), \forall y.$$  

![Graph of a convex function](image)
Sub-Gradients and Sub-Differentials

Recall that for differentiable convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

A vector \( d \) is a subgradient of a convex function \( f \) at \( x \) if

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Recall that for *differentiable* convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

A vector \( d \) is a *subgradient* of a convex function \( f \) at \( x \) if

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for \textit{differentiable} convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

\textbf{A vector} \( d \) \textbf{is a subgradient of a convex function} \( f \) \textbf{at} \( x \) \textbf{if}

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for differentiable convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

A vector \( d \) is a subgradient of a convex function \( f \) at \( x \) if

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for \textit{differentiable} convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

\textit{A vector} \( d \) \textit{is a subgradient of a convex function} \( f \) \textit{at} \( x \) \textit{if}

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for *differentiable* convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

A vector \( d \) is a *subgradient* of a convex function \( f \) at \( x \) if

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for \textit{differentiable} convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

\textit{A vector} \( d \) \textit{is a subgradient of a convex function} \( f \) \textit{at} \( x \) \textit{if}

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]
Sub-Gradients and Sub-Differentials

Recall that for \textit{differentiable} convex functions we have

\[ f(y) \geq f(x) + \nabla f(x)^T (y - x), \forall x, y. \]

\textit{A vector} \( d \) \textit{is a subgradient of a convex function} \( f \) \textit{at} \( x \) \textit{if}

\[ f(y) \geq f(x) + d^T (y - x), \forall y. \]

- \( f \) \textit{is differentiable at} \( x \) \textit{iff} \( \nabla f(x) \) \textit{is the only subgradient}.
- At non-differentiable \( x \), we have a set of subgradients.
- Set of subgradients is the \textit{sub-differential} \( \partial f(x) \).
- Note that \( 0 \in \partial f(x) \) \textit{iff} \( x \) \textit{is a global minimum}.
Sub-Differential of Absolute Value and Max Functions

- The sub-differential of the absolute value function:

\[
\partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases}
\]

(sign of the variable if non-zero, anything in \([-1, 1]\) at 0)
Sub-Differential of Absolute Value and Max Functions

- The sub-differential of the absolute value function:

\[ \partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases} \]

(sign of the variable if non-zero, anything in \([-1, 1]\) at 0)
Sub-Differential of Absolute Value and Max Functions

- The sub-differential of the absolute value function:

$$\partial |x| = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 \\
[-1, 1] & \text{if } x = 0
\end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)
The sub-differential of the absolute value function:

$$\partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases}$$

(sign of the variable if non-zero, anything in $[-1, 1]$ at 0)
The sub-differential of the absolute value function:

\[ \partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases} \]

(sign of the variable if non-zero, anything in \([-1, 1]\) at 0)
Sub-Differential of Absolute Value and Max Functions

The sub-differential of the absolute value function:

\[
\partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases}
\]

(sign of the variable if non-zero, anything in \([-1, 1] \text{ at } 0\))
Sub-Differential of Absolute Value and Max Functions

The sub-differential of the absolute value function:

\[ \partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases} \]

(sign of the variable if non-zero, anything in \([-1, 1]\) at 0)
Sub-Differential of Absolute Value and Max Functions

The sub-differential of the absolute value function:

\[
\partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0
\end{cases}
\]

(sign of the variable if non-zero, anything in \([-1, 1] \) at 0)
The sub-differential of the absolute value function:

\[
\partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases}
\]

(sign of the variable if non-zero, anything in \([-1, 1]\) at 0)
Sub-Differential of Absolute Value and Max Functions

- The sub-differential of the absolute value function:

\[ \partial |x| = \begin{cases} 
1 & x > 0 \\
-1 & x < 0 \\
[-1, 1] & x = 0 
\end{cases} \]

(sign of the variable if non-zero, anything in \([-1, 1]\) at 0)

- The sub-differential of the maximum of differentiable \(f_i\):

\[ \partial \max\{f_1(x), f_2(x)\} = \begin{cases} 
\nabla f_1(x) & f_1(x) > f_2(x) \\
\nabla f_2(x) & f_2(x) > f_1(x) \\
\theta \nabla f_1(x) + (1 - \theta) \nabla f_2(x) & f_1(x) = f_2(x) 
\end{cases} \]

(any convex combination of the gradients of the argmax)
Sub-gradient method

- The sub-gradient method:

\[ x^+ = x - \alpha d, \]

for some \( d \in \partial f(x) \).
Sub-gradient method

The sub-gradient method:

\[ x^+ = x - \alpha d, \]

for some \( d \in \partial f(x) \).

The steepest descent step is given by \( \arg \min_{d \in \partial f(x)} \|d\| \).

(often hard to compute, but easy for \( \ell_1 \)-regularization)
Sub-gradient method

- The sub-gradient method:
  \[ x^+ = x - \alpha d, \]
  for some \( d \in \partial f(x) \).
- The steepest descent step is given by \( \arg \min_{d \in \partial f(x)} \{ \|d\| \} \).
  (often hard to compute, but easy for \( \ell_1 \)-regularization)
- Otherwise, may increase the objective even for small \( \alpha \).
- But \( \|x^+ - x^*\| \leq \|x - x^*\| \) for small enough \( \alpha \).
- For convergence, we require \( \alpha \to 0 \).
Sub-gradient method

- The sub-gradient method:
  \[ x^+ = x - \alpha d, \]
  for some \( d \in \partial f(x) \).

- The steepest descent step is given by \( \arg\min_{d \in \partial f(x)} \{ \|d\| \} \).
  (often hard to compute, but easy for \( \ell_1 \)-regularization)

- Otherwise, may increase the objective even for small \( \alpha \).
- But \( \|x^+ - x^*\| \leq \|x - x^*\| \) for small enough \( \alpha \).
- For convergence, we require \( \alpha \to 0 \).
- Many variants average the iterations:
  \[ \bar{x}^k = \sum_{i=0}^{k-1} w_i x^i. \]

- Many variants average the gradients (‘dual averaging’):
  \[ \bar{d}^k = \sum_{i=0}^{k-1} w_i d^i. \]
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

Alternative is cutting-plane/bundle methods: Minimize an approximation based on all subgradients $\{d_t\}$. But have the same rates as the subgradient method. (tend to be better in practice)

Bad news: Rates are optimal for black-box methods. But, we often have more than a black-box: We can use structure to get faster rates than black-box methods.
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly-Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Strongly-Convex</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
</tbody>
</table>

Alternative is cutting-plane/bundle methods: Minimize an approximation based on all subgradients $\{d_t\}$.

But have the same rates as the subgradient method. (Tend to be better in practice)

Bad news: Rates are optimal for black-box methods. But, we often have more than a black-box: We can use structure to get faster rates than black-box methods.
**Convex Optimization Zoo**

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly-Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Strongly-Convex</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
</tbody>
</table>

- Alternative is cutting-plane/bundle methods:
  - Minimize an approximation based on *all* subgradients $\{d_t\}$.
  - But have the *same rates as the subgradient method*.
    (tend to be better in practice)
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly-Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Strongly-Convex</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
</tbody>
</table>

- Alternative is cutting-plane/bundle methods:
  - Minimize an approximation based on all subgradients $\{d_t\}$.
  - But have the same rates as the subgradient method.
    (tend to be better in practice)

- Bad news: Rates are optimal for black-box methods.
# Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly-Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Strongly-Convex</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
</tbody>
</table>

- Alternative is cutting-plane/bundle methods:
  - Minimize an approximation based on all subgradients $\{d_t\}$.
  - But have the same rates as the subgradient method.
  (tend to be better in practice)

- Bad news: Rates are optimal for black-box methods.
- But, we often have more than a black-box:
  - We can use structure to get faster rates than black-box methods.
Smoothing Approximations of Non-Smooth Functions

- Smoothing: replace non-smooth $f$ with smooth $f_\epsilon$.
- Apply a fast method for smooth optimization.
Smoothing Approximations of Non-Smooth Functions

- Smoothing: replace non-smooth $f$ with smooth $f_\epsilon$.
- Apply a fast method for smooth optimization.
- Smooth approximation to the absolute value:

$$|x| \approx \sqrt{x^2 + \nu}.$$
Smoothing Approximations of Non-Smooth Functions

- Smoothing: replace non-smooth $f$ with smooth $f_\varepsilon$.
- Apply a fast method for smooth optimization.
- Smooth approximation to the absolute value:

$$ |x| \approx \sqrt{x^2 + \nu}. $$
Smoothing Approximations of Non-Smooth Functions

- Smoothing: replace non-smooth $f$ with smooth $f_\varepsilon$.
- Apply a fast method for smooth optimization.

Smooth approximation to the absolute value:

$$|x| \approx \sqrt{x^2 + \nu}.$$ 

Smooth approximation to the max function:

$$\max\{a, b\} \approx \log(\exp(a) + \exp(b))$$

Smooth approximation to the hinge loss:

$$\max\{0, x\} \approx \begin{cases} 
0 & x \geq 1 \\
1 - x^2 & t < x < 1 \\
(1 - t)^2 + 2(1 - t)(t - x) & x \leq t
\end{cases}$$

Generic smoothing strategy: strongly-convex regularization of convex conjugate. 

[Nesterov, 2005]
Smoothing Approximations of Non-Smooth Functions

- Smoothing: replace non-smooth $f$ with smooth $f_{\epsilon}$.
- Apply a fast method for smooth optimization.
- Smooth approximation to the absolute value:
  $$ |x| \approx \sqrt{x^2 + \nu}. $$

- Smooth approximation to the max function:
  $$ \max\{a, b\} \approx \log(\exp(a) + \exp(b)) $$

- Smooth approximation to the hinge loss:
  $$ \max\{0, x\} \approx \begin{cases} 
  0 & x \geq 1 \\
  1 - x^2 & t < x < 1 \\
  (1 - t)^2 + 2(1 - t)(t - x) & x \leq t 
  \end{cases} $$

- Generic smoothing strategy: strongly-convex regularization of convex conjugate. [Nesterov, 2005]
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
</tbody>
</table>

You can get the $O(1/t)$ rate for $\min_x \max_i \{f_i(x)\}$ for $f_i$ convex and smooth using the mirror-prox method. 

[Nemirovski, 2004] See also Chambolle & Pock [2010].
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Smoothed to $1/\varepsilon$, Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
</tbody>
</table>

Smoothing is only faster if you use Nesterov’s method. In practice, faster to slowly decrease smoothing level and use L-BFGS or non-linear CG.

You can get the $O(1/t)$ rate for $\min x \max_i \{f_i(x)\}$ for $f_i$ convex and smooth using mirror-prox method. ([Nemirovski, 2004](#))

See also Chambolle & Pock [2010](#).
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

- Smoothing is only faster if you use Nesterov's method.

In practice, faster to slowly decrease smoothing level and use L-BFGS or non-linear CG. You can get the $O(1/t)$ rate for $\min_x \max_i \{f_i(x)\}$ for $f_i$ convex and smooth using mirror-prox method. [Nemirovski, 2004]

See also Chambolle & Pock [2010].
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

- Smoothing is only faster if you use Nesterov’s method.
- In practice, faster to slowly decrease smoothing level and use L-BFGS or non-linear CG.
Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Nesterov</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

- Smoothing is only faster if you use Nesterov’s method.
- In practice, faster to slowly decrease smoothing level and use L-BFGS or non-linear CG.
- You can get the $O(1/t)$ rate for $\min_x \max\{f_i(x)\}$ for $f_i$ convex and smooth using mirror-prox method.[Nemirovski, 2004]
  - See also Chambolle & Pock [2010].
Converting to Constrained Optimization

- Re-write non-smooth problem as constrained problem.
Converting to Constrained Optimization

- Re-write non-smooth problem as constrained problem.
- The problem
  \[ \min_x f(x) + \lambda \|x\|_1, \]
  is equivalent to the problem
  \[ \min_{x^+ \geq 0, x^- \geq 0} f(x^+ - x^-) + \lambda \sum_i (x_i^+ + x_i^-), \]

  or the problems
  \[ \min_{-y \leq x \leq y} f(x) + \lambda \sum_i y_i, \quad \min_{\|x\|_1 \leq \gamma} f(x) + \lambda \gamma \]
Converting to Constrained Optimization

- Re-write non-smooth problem as constrained problem.

The problem

$$\min_x f(x) + \lambda \|x\|_1,$$

is equivalent to the problem

$$\min_{x^+ \geq 0, x^- \geq 0} f(x^+ - x^-) + \lambda \sum_i (x_i^+ + x_i^-),$$

or the problems

$$\min_{-y \leq x \leq y} f(x) + \lambda \sum_i y_i, \quad \min_{\|x\|_1 \leq \gamma} f(x) + \lambda \gamma$$

- These are smooth objective with ‘simple’ constraints.
Recall: gradient descent minimizes quadratic approximation:

\[ x^+ = \arg \min_y \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 \right\}. \]
Recall: gradient descent minimizes quadratic approximation:

\[ x^+ = \arg \min_y \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 \right\}. \]

Consider minimizing subject to simple constraints:

\[ x^+ = \arg \min_{y \in C} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 \right\}. \]
Optimization with Simple Constraints

- Recall: gradient descent minimizes quadratic approximation:

\[ x^+ = \arg \min_y \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \| y - x \|^2 \right\}. \]

- Consider minimizing subject to simple constraints:

\[ x^+ = \arg \min_{y \in C} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \| y - x \|^2 \right\}. \]

- Equivalent to projection of gradient descent:

\[ x^{GD} = x - \alpha \nabla f(x), \]

\[ x^+ = \arg \min_{y \in C} \left\{ \| y - x^{GD} \| \right\}. \]
Gradient Projection

\[ f(x) \]
Gradient Projection
Gradient Projection
Gradient Projection

- Convex Functions
- Smooth Optimization
- Non-Smooth Optimization
- Randomized Algorithms
- Parallel/Distributed Optimization

Feasible Set

\[ f(x) \]

\[ x_k \]

\[ x_1 \]

\[ x_2 \]
Gradient Projection

\[ f(x) \]

Feasible Set

\[ x^k - \alpha \nabla f(x^k) \]
Gradient Projection

\[ f(x) \]

Feasible Set

\[ x^k - \alpha \nabla f(x^k) \]
Gradient Projection

\[ f(x) \]

Feasible Set

\[ x^k - \alpha \nabla f(x^k) \]

\[ [x^k - \alpha \nabla f(x^k)]^+ \]
Projection Onto Simple Sets

Projections onto simple sets:

- $\arg\min_{y \geq 0} \|y - x\| = \max\{x, 0\}$
Projection Onto Simple Sets

Projections onto simple sets:

- $\arg \min_{y \geq 0} \|y - x\| = \max\{x, 0\}$
- $\arg \min_{l \leq y \leq u} \|y - x\| = \max\{l, \min\{x, u\}\}$
Projection Onto Simple Sets

Projections onto simple sets:

- \( \arg\min_{y \geq 0} \|y - x\| = \max\{x, 0\} \)
- \( \arg\min_{l \leq y \leq u} \|y - x\| = \max\{l, \min\{x, u\}\} \)
- \( \arg\min_{a^T y = b} \|y - x\| = x + (b - a^T x)a/\|a\|^2 \).
Projection Onto Simple Sets

Projections onto simple sets:

- \( \arg\min_{y \geq 0} \|y - x\| = \max\{x, 0\} \)
- \( \arg\min_{l \leq y \leq u} \|y - x\| = \max\{l, \min\{x, u\}\} \)
- \( \arg\min_{a^Ty = b} \|y - x\| = x + (b - a^Tx)a/\|a\|^2 \)
- \( \arg\min_{a^Ty \geq b} \|y - x\| = \begin{cases} x & a^Tx \geq b \\ x + (b - a^Tx)a/\|a\|^2 & a^Tx < b \end{cases} \)
Projections onto simple sets:

- \( \arg \min_{y \geq 0} \| y - x \| = \max \{x, 0\} \)
- \( \arg \min_{l \leq y \leq u} \| y - x \| = \max \{l, \min\{x, u\}\} \)
- \( \arg \min_{a^T y = b} \| y - x \| = x + (b - a^T x) a / \| a \|^2 \)
- \( \arg \min_{a^T y \geq b} \| y - x \| = \begin{cases} x & a^T x \geq b \\ x + (b - a^T x) a / \| a \|^2 & a^T x < b \end{cases} \)
- \( \arg \min_{\| y \| \leq \tau} \| y - x \| = \tau x / \| x \| . \)
Projection Onto Simple Sets

Projections onto simple sets:

- \( \text{arg min}_{y \geq 0} \|y - x\| = \max\{x, 0\} \)
- \( \text{arg min}_{l \leq y \leq u} \|y - x\| = \max\{l, \min\{x, u\}\} \)
- \( \text{arg min}_{a^Ty = b} \|y - x\| = x + (b - a^Tx)a/\|a\|^2. \)
- \( \text{arg min}_{a^Ty \geq b} \|y - x\| = \begin{cases} x & a^Tx \geq b \\ x + (b - a^Tx)a/\|a\|^2 & a^Tx < b \end{cases} \)
- \( \text{arg min}_{\|y\| \leq \tau} \|y - x\| = \tau x/\|x\|. \)
- Linear-time algorithm for \( \ell_1 \)-norm \( \|y\|_1 \leq \tau. \)
Projection Onto Simple Sets

Projections onto simple sets:

- \( \arg \min_{y \geq 0} \| y - x \| = \max\{ x, 0 \} \)
- \( \arg \min_{l \leq y \leq u} \| y - x \| = \max\{ l, \min\{ x, u \} \} \)
- \( \arg \min_{a^T y = b} \| y - x \| = x + (b - a^T x)a / \| a \|^2 \)
- \( \arg \min_{a^T y \geq b} \| y - x \| = \begin{cases} x & a^T x \geq b \\ x + (b - a^T x)a / \| a \|^2 & a^T x < b \end{cases} \)
- \( \arg \min_{\| y \| \leq \tau} \| y - x \| = \tau x / \| x \| . \)
- Linear-time algorithm for \( \ell_1 \)-norm \( \| y \|_1 \leq \tau \).
- Linear-time algorithm for probability simplex \( y \geq 0, \sum y = 1 \).
Projection Onto Simple Sets

Projections onto simple sets:

- \( \arg\min_{y \geq 0} \| y - x \| = \max\{x, 0\} \)
- \( \arg\min_{l \leq y \leq u} \| y - x \| = \max\{l, \min\{x, u\}\} \)
- \( \arg\min_{a^T y = b} \| y - x \| = x + (b - a^T x) a / \| a \|^2 \)
- \( \arg\min_{a^T y \geq b} \| y - x \| = \begin{cases} x & a^T x \geq b \\ x + (b - a^T x) a / \| a \|^2 & a^T x < b \end{cases} \)
- \( \arg\min_{\| y \| \leq \tau} \| y - x \| = \tau x / \| x \| \).
- Linear-time algorithm for \( l_1 \)-norm \( \| y \|_1 \leq \tau \).
- Linear-time algorithm for probability simplex \( y \geq 0, \sum y = 1 \).
- Intersection of simple sets: Dykstra’s algorithm.
### Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(Subgradient)</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>P(Subgradient)</td>
<td>Strongly</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>P(Nesterov)</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>P(Gradient)</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>P(Nesterov)</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>P(Gradient)</td>
<td>Strongly</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
<tr>
<td>P(Nesterov)</td>
<td>Strongly</td>
<td>$O((1 - \sqrt{\mu/L})^t)$</td>
</tr>
<tr>
<td>P(Newton)</td>
<td>Strongly</td>
<td>$O(\prod_{i=1}^t \rho_t)$, $\rho_t \to 0$</td>
</tr>
</tbody>
</table>

- Convergence rates are the same for projected versions!
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(Subgradient)</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>P(Subgradient)</td>
<td>Strongly</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>P(Nesterov)</td>
<td>Smoothed to $1/\epsilon$, Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>P(Gradient)</td>
<td>Convex</td>
<td>$O(1/t)$</td>
</tr>
<tr>
<td>P(Nesterov)</td>
<td>Convex</td>
<td>$O(1/t^2)$</td>
</tr>
<tr>
<td>P(Gradient)</td>
<td>Strongly</td>
<td>$O((1 - \mu/L)^t)$</td>
</tr>
<tr>
<td>P(Nesterov)</td>
<td>Strongly</td>
<td>$O((1 - \sqrt{\mu/L})^t)$</td>
</tr>
<tr>
<td>P(Newton)</td>
<td>Strongly</td>
<td>$O(\prod_{i=1}^{t} \rho_t), \rho_t \to 0$</td>
</tr>
</tbody>
</table>

- Convergence rates are the same for projected versions!
- Can do many of the same tricks (i.e. Armijo line-search, polynomial interpolation, Barzilai-Borwein, quasi-Newton).
# Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(\text{Subgradient}) )</td>
<td>Convex</td>
<td>( O(1/\sqrt{t}) )</td>
</tr>
<tr>
<td>( P(\text{Subgradient}) )</td>
<td>Strongly</td>
<td>( O(1/t) )</td>
</tr>
<tr>
<td>( P(\text{Nesterov}) )</td>
<td>Smoothed to ( 1/\epsilon ), Convex</td>
<td>( O(1/t) )</td>
</tr>
<tr>
<td>( P(\text{Gradient}) )</td>
<td>Convex</td>
<td>( O(1/t) )</td>
</tr>
<tr>
<td>( P(\text{Nesterov}) )</td>
<td>Convex</td>
<td>( O(1/t^2) )</td>
</tr>
<tr>
<td>( P(\text{Gradient}) )</td>
<td>Strongly</td>
<td>( O((1 - \mu/L)^t) )</td>
</tr>
<tr>
<td>( P(\text{Nesterov}) )</td>
<td>Strongly</td>
<td>( O((1 - \sqrt{\mu/L})^t) )</td>
</tr>
<tr>
<td>( P(\text{Newton}) )</td>
<td>Strongly</td>
<td>( O(\prod_{i=1}^{t} \rho_t), \rho_t \to 0 )</td>
</tr>
</tbody>
</table>

- Convergence rates are the same for projected versions!
- Can do many of the same tricks (i.e. Armijo line-search, polynomial interpolation, Barzilai-Borwein, quasi-Newton).
- For Newton, you need to project under \( \| \cdot \| \nabla^2 f(x) \) (expensive, but special tricks for the case of simplex or lower/upper bounds).
- You don’t need to compute the projection exactly.
A generalization of projected-gradient is **Proximal-gradient**.
The proximal-gradient method addresses problem of the form

$$\min_x f(x) + r(x),$$

where $f$ is smooth but $r$ is a general convex function.
Proximal-Gradient Method

- A generalization of projected-gradient is **Proximal-gradient**.
- The proximal-gradient method addresses problem of the form

  \[
  \min_{x} f(x) + r(x),
  \]

  where \( f \) is smooth but \( r \) is a general convex function.
- Applies **proximity** operator of \( r \) to gradient descent on \( f \):

  \[
  x^{GD} = x - \alpha \nabla f(x),
  \]

  \[
  x^{+} = \arg \min_{y} \left\{ \frac{1}{2} \| y - x^{GD} \|^2 + \alpha r(y) \right\},
  \]
Proximal-Gradient Method

- A generalization of projected-gradient is **Proximal-gradient**.
- The proximal-gradient method addresses problem of the form

\[
\min_x f(x) + r(x),
\]

where \( f \) is smooth but \( r \) is a general convex function.
- Applies **proximity** operator of \( r \) to gradient descent on \( f \):

\[
x^{GD} = x - \alpha \nabla f(x),
\]

\[
x^+ = \arg \min_y \left\{ \frac{1}{2} \|y - x^{GD}\|^2 + \alpha r(y) \right\},
\]

- Equivalent to using the approximation

\[
x^+ = \arg \min_y \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 + r(y) \right\}.
\]

- **Convergence rates are still the same as for minimizing \( f \).**
The proximal operator is the solution to

\[ \text{prox}_r[y] = \arg \min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2} \| x - y \|^2. \]
The proximal operator is the solution to

\[
\text{prox}_r[y] = \arg \min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2} \|x - y\|^2.
\]

For L1-regularization, we obtain iterative soft-thresholding:

\[
x^+ = \text{softThresh}_{\alpha \lambda}[x - \alpha \nabla f(x)].
\]
Proximal Operator, Iterative Soft Thresholding

- The proximal operator is the solution to

\[
\text{prox}_r[y] = \arg \min_{x \in \mathbb{R}^P} \ r(x) + \frac{1}{2} \|x - y\|^2.
\]

- For L1-regularization, we obtain iterative soft-thresholding:

\[
x^+ = \text{softThresh}_{\alpha \lambda}[x - \alpha \nabla f(x)].
\]

- Example with \( \lambda = 1 \):

\[
\begin{bmatrix}
0.6715 \\
-1.2075 \\
0.7172 \\
1.6302 \\
0.4889
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.6715 \\
-1.2075 \\
0.7172 \\
0.4889
\end{bmatrix}
\]
Proximal Operator, Iterative Soft Thresholding

- The **proximal operator** is the solution to

$$\text{prox}_r[y] = \arg \min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2} \|x - y\|^2.$$ 

- For L1-regularization, we obtain **iterative soft-thresholding**:

$$x^+ = \text{softThresh}_{\alpha \lambda}[x - \alpha \nabla f(x)].$$

- Example with $\lambda = 1$:

<table>
<thead>
<tr>
<th>Input</th>
<th>Threshold</th>
<th>Soft-Threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 0.6715 \ -1.2075 \ 0.7172 \ 1.6302 \ 0.4889 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ -1.2075 \ 0 \ 1.6302 \ 0 \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>
**Proximal Operator, Iterative Soft Thresholding**

- The **proximal operator** is the solution to

$$\text{prox}_r[y] = \arg \min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2}\|x - y\|^2.$$ 

- For L1-regularization, we obtain **iterative soft-thresholding**:

$$x^+ = \text{softThresh}_{\alpha \lambda}[x - \alpha \nabla f(x)].$$

- Example with $\lambda = 1$:

<table>
<thead>
<tr>
<th>Input</th>
<th>Threshold</th>
<th>Soft-Threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \begin{bmatrix} 0.6715 \ -1.2075 \ 0.7172 \ 1.6302 \ 0.4889 \end{bmatrix} ]</td>
<td>[ \begin{bmatrix} 0 \ -1.2075 \ 0 \ 1.6302 \ 0 \end{bmatrix} ]</td>
<td>[ \begin{bmatrix} 0 \ -0.2075 \ 0 \ 0.6302 \ 0 \end{bmatrix} ]</td>
</tr>
</tbody>
</table>
Special case of Projected-Gradient Methods

- **Projected-gradient methods** are another special case:

\[
r(x) = \begin{cases} 
0 & \text{if } x \in C \\
\infty & \text{if } x \notin C
\end{cases}
\]
Special case of Projected-Gradient Methods

- **Projected-gradient** methods are another special case:

\[
    r(x) = \begin{cases} 
        0 & \text{if } x \in C \\
        \infty & \text{if } x \notin C
    \end{cases},
\]

which gives

\[
    x^+ = \text{project}_C[x - \alpha \nabla f(x)],
\]
**Special case of Projected-Gradient Methods**

- **Projected-gradient** methods are another special case:

\[
r(x) = \begin{cases} 
0 & \text{if } x \in \mathcal{C} \\
\infty & \text{if } x \notin \mathcal{C}
\end{cases},
\]

\[
x^+ = \text{project}_C[x - \alpha \nabla f(x)],
\]
Special case of Projected-Gradient Methods

- **Projected-gradient** methods are another special case:

\[
r(x) = \begin{cases} 
0 & \text{if } x \in \mathcal{C} \\
\infty & \text{if } x \notin \mathcal{C}
\end{cases}
\]

\[x^+ = \text{project}_\mathcal{C}[x - \alpha \nabla f(x)],\]
Special case of Projected-Gradient Methods

- **Projected-gradient methods** are another special case:

\[
r(x) = \begin{cases} 
0 & \text{if } x \in C \\
\infty & \text{if } x \notin C 
\end{cases},
\]

gives

\[
x^+ = \text{project}_C[x - \alpha \nabla f(x)],
\]
Special case of Projected-Gradient Methods

- **Projected-gradient** methods are another special case:

\[
    r(x) = \begin{cases} 
        0 & \text{if } x \in C \\ 
        \infty & \text{if } x \notin C 
    \end{cases},
\]

gives

\[
    x^+ = \text{project}_C[x - \alpha \nabla f(x)],
\]
Special case of Projected-Gradient Methods

Projected-gradient methods are another special case:

\[ r(x) = \begin{cases} 
0 & \text{if } x \in C \\
\infty & \text{if } x \notin C
\end{cases}, \]

gives

\[ x^+ = \text{project}_C[x - \alpha \nabla f(x)], \]
Exact Proximal-Gradient Methods

- For what problems can we apply these methods?
Exact Proximal-Gradient Methods

- For what problems can we apply these methods?
- We can efficiently compute the proximity operator for:
  1. L1-Regularization.
Exact Proximal-Gradient Methods

- For what problems can we apply these methods?
- We can efficiently compute the proximity operator for:
  1. L1-Regularization.
  2. Group $\ell_1$-Regularization.
For what problems can we apply these methods?

We can efficiently compute the proximity operator for:

1. L1-Regularization.
2. Group $\ell_1$-Regularization.
3. Lower and upper bounds.
Exact Proximal-Gradient Methods

For what problems can we apply these methods?

We can efficiently compute the proximity operator for:

1. L1-Regularization.
2. Group $\ell_1$-Regularization.
3. Lower and upper bounds.
4. Small number of linear constraint.
5. Probability constraints.
6. A few other simple regularizers/constraints.

Can solve these non-smooth/constrained problems as fast as smooth/unconstrained problems!
Exact Proximal-Gradient Methods

For what problems can we apply these methods?

We can efficiently compute the proximity operator for:

1. L1-Regularization.
2. Group $\ell_1$-Regularization.
3. Lower and upper bounds.
4. Small number of linear constraint.
5. Probability constraints.
Exact Proximal-Gradient Methods

For what problems can we apply these methods?

- We can efficiently compute the proximity operator for:
  1. L1-Regularization.
  2. Group $l_1$-Regularization.
  3. Lower and upper bounds.
  4. Small number of linear constraint.
  5. Probability constraints.
  6. A few other simple regularizers/constraints.

Can solve these non-smooth/constrained problems as fast as smooth/unconstrained problems!

But for many problems we can not efficiently compute this operator.
Exact Proximal-Gradient Methods

For what problems can we apply these methods?

- We can efficiently compute the proximity operator for:
  1. L1-Regularization.
  2. Group $\ell_1$-Regularization.
  3. Lower and upper bounds.
  4. Small number of linear constraint.
  5. Probability constraints.
  6. A few other simple regularizers/constraints.

- Can solve these non-smooth/constrained problems as fast as smooth/unconstrained problems!
Exact Proximal-Gradient Methods

- For what problems can we apply these methods?
- We can efficiently compute the proximity operator for:
  1. L1-Regularization.
  2. Group $\ell_1$-Regularization.
  3. Lower and upper bounds.
  4. Small number of linear constraint.
  5. Probability constraints.
  6. A few other simple regularizers/constraints.
- Can solve these non-smooth/constrained problems as fast as smooth/unconstrained problems!
- But for many problems we can not efficiently compute this operator.
Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximity operator for:
  1. Structured sparsity.
  2. Penalties on the differences between variables.
  3. Regularizers and constraints on the singular values of matrices.
  4. Sums of simple functions.

Many recent works use inexact proximal-gradient methods:
Cai et al. [2010], Liu & Ye [2010], Barbero & Sra [2011], Fadili & Peyr´e [2011], Ma et al. [2011].

Do inexact methods have the same rates?
Yes, if the errors are appropriately controlled [Schmidt et al., 2011].
Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximity operator for:
  1. Structured sparsity.
Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximity operator for:
  1. Structured sparsity.
  2. Penalties on the differences between variables.
Inexact Proximal-Gradient Methods

We can efficiently approximate the proximity operator for:

1. Structured sparsity.
2. Penalties on the differences between variables.
3. Regularizers and constraints on the singular values of matrices.
Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximity operator for:
  1. Structured sparsity.
  2. Penalties on the differences between variables.
  3. Regularizers and constraints on the singular values of matrices.
  4. Sums of simple functions.
Inexact Proximal-Gradient Methods

- We can efficiently **approximate** the proximity operator for:
  1. **Structured sparsity**.
  2. Penalties on the differences between variables.
  3. Regularizers and constraints on the singular values of matrices.
  4. Sums of simple functions.

- Many recent works use **inexact proximal-gradient** methods:
  Cai et al. [2010], Liu & Ye [2010], Barbero & Sra [2011], Fadili & Peyré [2011], Ma et al. [2011]
Inexact Proximal-Gradient Methods

- We can efficiently **approximate** the proximity operator for:
  1. Structured sparsity.
  2. Penalties on the differences between variables.
  3. Regularizers and constraints on the singular values of matrices.
  4. Sums of simple functions.

- Many recent works use **inexact proximal-gradient** methods:
  Cai et al. [2010], Liu & Ye [2010], Barbero & Sra [2011], Fadili & Peyré [2011], Ma et al. [2011]

- Do **inexact methods** have the same rates?
Inexact Proximal-Gradient Methods

- We can efficiently approximate the proximity operator for:
  1. Structured sparsity.
  2. Penalties on the differences between variables.
  3. Regularizers and constraints on the singular values of matrices.
  4. Sums of simple functions.

- Many recent works use inexact proximal-gradient methods:
  Cai et al. [2010], Liu & Ye [2010], Barbero & Sra [2011], Fadili & Peyré [2011], Ma et al. [2011]

- Do inexact methods have the same rates?
  - Yes, if the errors are appropriately controlled. [Schmidt et al., 2011]
Proposition [Schmidt et al., 2011] If the sequences of gradient errors \( \|e_t\| \) and proximal errors \( \sqrt{\varepsilon_t} \) are in \( \{O((1 - \mu/L)^t)\} \), then the inexact proximal-gradient method has an error of \( O((1 - \mu/L)^t) \).
Convergence Rate of Inexact Proximal-Gradient

**Proposition** [Schmidt et al., 2011] If the sequences of gradient errors \( \{ ||e_t|| \} \) and proximal errors \( \{ \sqrt{\varepsilon_t} \} \) are in \( \{ O((1 - \mu/L)^t) \} \), then the inexact proximal-gradient method has an error of \( O((1 - \mu/L)^t) \).

- Classic result as a special case (constants are good).
- The rates degrades gracefully if the errors are larger.
- Similar analyses in convex case.
- **Huge improvement in practice** over black-box methods.
- Also exist accelerated and spectral proximal-gradient methods.
Discussion of Proximal-Gradient

- **Theoretical justification** for what works in practice.
- Significantly **extends class of tractable problems**.
- Many **applications** with inexact proximal operators:
  - Genomic expression, model predictive control, neuroimaging, satellite image fusion, simulating flow fields.
Discussion of Proximal-Gradient

- **Theoretical justification** for what works in practice.
- Significantly extends class of tractable problems.
- Many applications with inexact proximal operators:
  - Genomic expression, model predictive control, neuroimaging, satellite image fusion, simulating flow fields.

But, it assumes computing $\nabla f(x)$ and $\text{prox}_h[x]$ have similar cost.
Discussion of Proximal-Gradient

- **Theoretical justification** for what works in practice.
- Significantly **extends class of tractable problems**.
- Many **applications** with inexact proximal operators:
  - Genomic expression, model predictive control, neuroimaging, satellite image fusion, simulating flow fields.

- But, **it assumes computing $\nabla f(x)$ and $\text{prox}_h[x]$ have similar cost**.
- Often $\nabla f(x)$ is much more expensive:
  - We may have a large dataset.
  - Data-fitting term might be complex.
- Particularly true for **structured output prediction**:
  - Text, biological sequences, speech, images, matchings, graphs.
Costly Data-Fitting Term, Simple Regularizer

Consider fitting a conditional random field with \( \ell_1 \)-regularization:

\[
\min_{x \in \mathbb{R}^P} \quad \frac{1}{N} \sum_{i=1}^{N} f_i(x) + r(x)
\]

costly smooth + simple
Costly Data-Fitting Term, Simple Regularizer

Consider fitting a conditional random field with \( \ell_1 \)-regularization:

\[
\min_{x \in \mathbb{R}^P} \frac{1}{N} \sum_{i=1}^{N} f_i(x) + r(x)
\]

costly smooth + simple

Different than classic optimization (like linear programming).
(cheap smooth plus complex non-smooth)
Consider fitting a conditional random field with $\ell_1$-regularization:

$$\min_{x \in \mathbb{R}^P} \frac{1}{N} \sum_{i=1}^{N} f_i(x) + r(x)$$

costly smooth + simple

Different than classic optimization (like linear programming).

(cheap smooth plus complex non-smooth)

Inspiration from the smooth case:

- For smooth high-dimensional problems, L-BFGS quasi-Newton algorithm outperforms accelerated/spectral gradient methods.
Quasi-Newton Methods

- Gradient method for optimizing a smooth $f$:

$$x^+ = x - \alpha \nabla f(x).$$

$H$ approximates the second-derivative matrix.

L-BFGS is a particular strategy to choose the $H$ values:

Based on gradient differences.

Linear storage and linear time.

http://www.di.ens.fr/~mschmidt/Software/minFunc.html
Quasi-Newton Methods

- Gradient method for optimizing a smooth $f$:
  \[ x^+ = x - \alpha \nabla f(x). \]

- Newton-like methods alternatively use:
  \[ x^+ = x - \alpha H^{-1} \nabla f(x). \]

- $H$ approximates the second-derivative matrix.

[L-BFGS is a particular strategy to choose the $H$ values:
Based on gradient differences.
Linear storage and linear time.
http://www.di.ens.fr/~mschmidt/Software/minFunc.html]
Quasi-Newton Methods

- Gradient method for optimizing a smooth $f$:
  $$x^+ = x - \alpha \nabla f(x).$$

- Newton-like methods alternatively use:
  $$x^+ = x - \alpha H^{-1} \nabla f(x).$$

- $H$ approximates the second-derivative matrix.
- L-BFGS is a particular strategy to choose the $H$ values:
  - Based on gradient differences.
  - Linear storage and linear time.

http://www.di.ens.fr/~mschmidt/Software/minFunc.html
Gradient Method and Newton’s Method

\[ f(x) \]
Gradient Method and Newton’s Method
Gradient Method and Newton’s Method
Gradient Method and Newton’s Method

\[ \mathbf{Q}(\mathbf{x}) \]

\[ \mathbf{x} \rightarrow \mathbf{x} - \alpha \mathbf{f}'(\mathbf{x}) \]
Gradient Method and Newton’s Method

\[ f(x) \]

\[ x^k - \alpha H^{-1} f'(x) \]

\[ Q(x) \]

\[ x - \alpha f''(x) \]
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:

\[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the Newton-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the Newton-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- NO!
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the **Newton**-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- **NO!**
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r} [x - \alpha \nabla f(x)]. \]

- Can we just plug in the \textbf{Newton}-like step?
  \[ x^+ = \text{prox}_{\alpha r} [x - \alpha H^{-1} \nabla f(x)]. \]

- \textbf{NO!}
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the Newton-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- NO!
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the **Newton**-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- **NO!**
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r} [x - \alpha \nabla f(x)]. \]

- Can we just plug in the \textbf{Newton}-like step?
  \[ x^+ = \text{prox}_{\alpha r} [x - \alpha H^{-1} \nabla f(x)]. \]

- \textbf{NO!}
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the Newton-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- NO!
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the *Newton*-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1}\nabla f(x)]. \]

- **NO!**
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the *Newton*-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- NO!
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r} [x - \alpha \nabla f(x)]. \]

- Can we just plug in the Newton-like step?
  \[ x^+ = \text{prox}_{\alpha r} [x - \alpha H^{-1} \nabla f(x)]. \]

- NO!
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the \textbf{Newton}-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- \textbf{NO!}
Naive Proximal Quasi-Newton Method

- Proximal-gradient method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha \nabla f(x)]. \]

- Can we just plug in the Newton-like step?
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]. \]

- NO!
Two-Metric (Sub)Gradient Projection

- In some cases, we can modify $H$ to make this work:
  - Bound constraints.
  - Probability constraints.
  - L1-regularization.

- **Two-metric (sub)gradient projection.**

- Key idea: make $H$ diagonal with respect to coordinates near non-differentiability.
Comparing to accelerated/spectral/diagonal gradient

Comparing to methods that do not use L-BFGS (sido data):

http://www.di.ens.fr/~mschmidt/Software/L1General.html
Inexact Proximal-Newton

- The broken proximal-Newton method:

\[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1}\nabla f(x)], \]

with the Euclidean proximal operator:

\[ \text{prox}_r[y] = \arg \min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2}\|x - y\|^2, \]
Inexact Proximal-Newton

- The **fixed** proximal-Newton method:

\[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]_H, \]

with the Euclidean proximal operator:

\[ \text{prox}_r[y] = \arg\min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2} \|x - y\|^2, \]
Inexact Proximal-Newton

- The fixed proximal-Newton method:

  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1}\nabla f(x)]_H, \]

  with the non-Euclidean proximal operator:

  \[ \text{prox}_r[y]_H = \arg \min_{x \in \mathbb{R}^p} r(x) + \frac{1}{2}\|x - y\|_H^2, \]

  where \( \|x\|_H^2 = x^T H x. \)
Inexact Proximal-Newton

- The fixed proximal-Newton method:
  \[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]_H, \]

with the non-Euclidean proximal operator:

\[ \text{prox}_r[y]_H = \arg \min_{x \in \mathbb{R}^p} r(x) + \frac{1}{2} \|x - y\|^2_H, \]

where \( \|x\|^2_H = x^T H x \).

- Non-smooth Newton-like method
- Same convergence properties as smooth case.
Inexact Proximal-Newton

- The fixed proximal-Newton method:

\[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]_H, \]

with the non-Euclidean proximal operator:

\[ \text{prox}_r[y]_H = \arg \min_{x \in \mathbb{R}^p} r(x) + \frac{1}{2} \|x - y\|_H^2, \]

where \( \|x\|_H^2 = x^T H x. \)

- Non-smooth Newton-like method
- Same convergence properties as smooth case.
- But, the prox is expensive even with a simple regularizer.
Inexact Proximal-Newton

- The **fixed** proximal-Newton method:

\[ x^+ = \text{prox}_{\alpha r}[x - \alpha H^{-1} \nabla f(x)]_H, \]

with the **non-Euclidean** proximal operator:

\[ \text{prox}_r[y]_H = \arg \min_{x \in \mathbb{R}^P} r(x) + \frac{1}{2} \|x - y\|_H^2, \]

where \( \|x\|_H^2 = x^T H x. \)

- **Non-smooth** Newton-like method
- **Same convergence properties as smooth case.**
- But, the **prox** is expensive even with a simple regularizer.
- **Solution**: use a cheap approximate solution.

(e.g., spectral proximal-gradient)
Inexact Projected Newton
Inexact Projected Newton
Inexact Projected Newton

\[ f(x) \]

\[ x - \alpha f'(x) \]

Feasible Set
Inexact Projected Newton

\[ f(x) \]

\[ x - \alpha f'(x) \]

Feasible Set
Inexact Projected Newton

The Inexact Projected Newton method is a variant of the Newton method for optimization problems with constraints. The method involves making a step in the direction of the negative gradient, scaled by a factor $\alpha$, and then projecting the result back onto the feasible set.

Mathematically, this is expressed as:

$$ x^{+} = \text{Proj}(x - \alpha f'(x)) $$

where $x$ is the current iterate, $f'(x)$ is the gradient of $f$ at $x$, and $\alpha$ is a step size parameter. The projection ensures that the next iterate remains within the feasible set defined by the constraints.
Inexact Projected Newton

\[ f(x) \]

Feasible Set
Inexact Projected Newton

$\mathbf{f}(\mathbf{x})$

$\mathbf{Q}(\mathbf{y})$

$\mathbf{y}^0 = \mathbf{x}$

Feasible Set
Inexact Projected Newton

\[ f(x) \]

\[ Q(y) \]

\[ y^0 - \alpha Q'(y^0) \]

\[ y^0 = x \]

Feasible Set

Inexact Projected Newton
Inexact Projected Newton
Inexact Projected Newton

\[ f(x) \]

\[ Q(y) \]

\[ y^0 = x \]

\[ y^1 \]

\[ y^2 \]

Feasible Set
Inexact Projected Newton

Feasible Set

\(f(x)\)

\(Q(y)\)

\(y^0 = x\)

\(y^1\)

\(y^2\)

\(y^3\)
Inexact Projected Newton
Inexact Projected Newton

\[ f(x) \]
\[ Q(y) \]
\[ x^+ = y^t \]
\[ y^0 = x \]
Feasible Set
Projected Quasi-Newton (PQN) Algorithm

- A proximal quasi-Newton (PQN) algorithm:
  [Schmidt et al., 2009, Schmidt, 2010]
A proximal quasi-Newton (PQN) algorithm:

[Schmidt et al., 2009, Schmidt, 2010]

- Outer: evaluate $f(x)$ and $\nabla f(x)$, use L-BFGS to update $H$. 

"Optimizing costly functions with simple constraints". 

"Optimizing costly functions with simple regularizers".
Projected Quasi-Newton (PQN) Algorithm

- A proximal quasi-Newton (PQN) algorithm:
  [Schmidt et al., 2009, Schmidt, 2010]
  - Outer: evaluate $f(x)$ and $\nabla f(x)$, use L-BFGS to update $H$.
  - Inner: spectral proximal-gradient to approximate proximal operator:
    - Requires multiplication by $H$ (linear-time for L-BFGS).
    - Requires proximal operator of $r$ (cheap for simple constraints).
Projected Quasi-Newton (PQN) Algorithm

A proximal quasi-Newton (PQN) algorithm:

[Schmidt et al., 2009, Schmidt, 2010]

- Outer: evaluate $f(x)$ and $\nabla f(x)$, use L-BFGS to update $H$.
- Inner: spectral proximal-gradient to approximate proximal operator:
  - Requires multiplication by $H$ (linear-time for L-BFGS).
  - Requires proximal operator of $r$ (cheap for simple constraints).
- For small $\alpha$, one iteration is sufficient to give descent.
Projected Quasi-Newton (PQN) Algorithm

- A proximal quasi-Newton (PQN) algorithm:
  [Schmidt et al., 2009, Schmidt, 2010]
  - Outer: evaluate $f(x)$ and $\nabla f(x)$, use L-BFGS to update $H$.
  - Inner: spectral proximal-gradient to approximate proximal operator:
    - Requires multiplication by $H$ (linear-time for L-BFGS).
    - Requires proximal operator of $r$ (cheap for simple constraints).
  - For small $\alpha$, one iteration is sufficient to give descent.
- Cheap inner iterations lead to fewer expensive outer iterations.
Projected Quasi-Newton (PQN) Algorithm

A proximal quasi-Newton (PQN) algorithm:
[Schmidt et al., 2009, Schmidt, 2010]

- Outer: evaluate $f(x)$ and $\nabla f(x)$, use L-BFGS to update $H$.
- Inner: spectral proximal-gradient to approximate proximal operator:
  - Requires multiplication by $H$ (linear-time for L-BFGS).
  - Requires proximal operator of $r$ (cheap for simple constraints).
  - For small $\alpha$, one iteration is sufficient to give descent.

Cheap inner iterations lead to fewer expensive outer iterations.

“Optimizing costly functions with simple constraints”.
A proximal quasi-Newton (PQN) algorithm:
[Schmidt et al., 2009, Schmidt, 2010]
- Outer: evaluate $f(x)$ and $\nabla f(x)$, use L-BFGS to update $H$.
- Inner: spectral proximal-gradient to approximate proximal operator:
  - Requires multiplication by $H$ (linear-time for L-BFGS).
  - Requires proximal operator of $r$ (cheap for simple constraints).
- For small $\alpha$, one iteration is sufficient to give descent.

Cheap inner iterations lead to fewer expensive outer iterations.

“Optimizing costly functions with simple constraints”.
“Optimizing costly functions with simple regularizers”.
Graphical Model Structure Learning with Groups

Comparing PQN to first-order methods on a graphical model structure learning problem. [Gasch et al., 2000, Duchi et al., 2008].

![Graphical Model Structure Learning with Groups](image-url)
Alternating Direction Method of Multipliers

**Alternating direction method of multipliers (ADMM)** solves:

$$\min_{Ax + By = c} f(x) + r(y).$$

- Alternate between prox-like operators with respect to $f$ and $r$.  

- If prox cannot be computed exactly: Linearized ADMM.
Alternating Direction Method of Multipliers

- Alternating direction method of multipliers (ADMM) solves:
  \[
  \min_{Ax+By=c} f(x) + r(y).
  \]

- Alternate between prox-like operators with respect to $f$ and $r$.
- Can introduce constraints to convert to this form:
  \[
  \min_x f(Ax) + r(x) \iff \min_{x=Ay} f(x) + r(y),
  \]
Alternating Direction Method of Multipliers

- **Alternating direction method of multipliers (ADMM)** solves:

\[
\min_{Ax+By=c} f(x) + r(y).
\]

- Alternate between prox-like operators with respect to \( f \) and \( r \).
- Can introduce constraints to convert to this form:

\[
\min_x f(Ax) + r(x) \quad \Leftrightarrow \quad \min_{x=Ay} f(x) + r(y),
\]

\[
\min_x f(x) + r(Bx) \quad \Leftrightarrow \quad \min_{y=Bx} f(x) + r(y).
\]
Alternating Direction Method of Multipliers

- **Alternating direction method of multipliers (ADMM)** solves:

  \[
  \min_{Ax+By=c} f(x) + r(y).
  \]

- Alternate between prox-like operators with respect to \( f \) and \( r \).
- Can **introduce constraints** to convert to this form:

  \[
  \min_x f(Ax) + r(x) \quad \Leftrightarrow \quad \min_{x=Ay} f(x) + r(y),
  \]

  \[
  \min_x f(x) + r(Bx) \quad \Leftrightarrow \quad \min_{y=Bx} f(x) + r(y).
  \]

- If prox can not be computed exactly: **Linearized ADMM**.
Dual Methods

- Strongly-convex problems have smooth duals.
- Solve the dual instead of the primal.
Dual Methods

- Strongly-convex problems have smooth duals.
- Solve the dual instead of the primal.
- SVM non-smooth strongly-convex primal:

\[
\min_{\mathbf{x}} C \sum_{i=1}^{N} \max\{0, 1 - b_i a_i^T \mathbf{x}\} + \frac{1}{2} \|\mathbf{x}\|^2.
\]

- SVM smooth dual:

\[
\min_{0 \leq \alpha \leq C} \frac{1}{2} \alpha^T A A^T \alpha - \sum_{i=1}^{N} \alpha_i
\]

- Smooth bound constrained problem:
  - Two-metric projection (efficient Newton-liked method).
  - Randomized coordinate descent (next section).
State of the art methods consider several other issues:

- **Shrinking**: Identify variables likely to stay zero. [El Ghaoui et al., 2010].
- **Continuation**: Start with a large $\lambda$ and slowly decrease it. [Xiao and Zhang, 2012]
- **Frank-Wolfe**: Using linear approximations to obtain efficient/sparse updates.
Frank-Wolfe Method

- In some cases the projected gradient step

\[ x^+ = \arg \min_{y \in C} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 \right\}, \]

may be hard to compute (e.g., dual of max-margin Markov networks).
Frank-Wolfe Method

- In some cases the projected gradient step

\[ x^+ = \arg \min_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \|y - x\|^2 \right\}, \]

may be hard to compute (e.g., dual of max-margin Markov networks).

- Frank-Wolfe method simply uses:

\[ x^+ = \arg \min_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) \right\}, \]

requires compact \( \mathcal{C} \), takes convex combination of \( x \) and \( x^+ \).
Frank-Wolfe Method

- In some cases the projected gradient step

\[ x^+ = \arg \min_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2 \alpha} \|y - x\|^2 \right\} , \]

may be hard to compute (e.g., dual of max-margin Markov networks).

- Frank-Wolfe method simply uses:

\[ x^+ = \arg \min_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) \right\} , \]

requires compact \( \mathcal{C} \), takes convex combination of \( x \) and \( x^+ \).

- Iterate can be written as convex combination of vertices of \( \mathcal{C} \).

- \( O(1/t) \) rate for smooth convex objectives, some linear convergence results for smooth and strongly-convex.[Jaggi, 2013]
Alternatives to Quadratic/Linear Surrogates

- **Mirror descent** uses the iterations [Beck & Teboulle, 2003]

\[
x^+ = \arg \min_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \mathcal{D}(x, y) \right\},
\]

where \( \mathcal{D} \) is a Bregman-divergence:

- \( \mathcal{D} = \|x - y\|^2 \) (gradient method).
- \( \mathcal{D} = \|x - y\|_H^2 \) (Newton’s method).
- \( \mathcal{D} = \sum_i x_i \log \left( \frac{x_i}{y_i} \right) - \sum_i (x_i - y_i) \) (exponentiated gradient).
Alternatives to Quadratic/Linear Surrogates

- **Mirror descent** uses the iterations [Beck & Teboulle, 2003]

\[ x^+ = \arg \min_{y \in C} \left\{ f(x) + \nabla f(x)^T (y - x) + \frac{1}{2\alpha} \mathcal{D}(x, y) \right\}, \]

where \( \mathcal{D} \) is a Bregman-divergence:

- \( \mathcal{D} = \|x - y\|^2 \) (gradient method).
- \( \mathcal{D} = \|x - y\|^2_H \) (Newton’s method).
- \( \mathcal{D} = \sum_i x_i \log\left(\frac{x_i}{y_i}\right) - \sum_i (x_i - y_i) \) (exponentiated gradient).

- **Mairal [2013,2014]** considers general surrogate optimization:

\[ x^+ = \arg \min_{y \in C} \{ g(y) \}, \]

where \( g \) upper bounds \( f \), \( g(x) = f(x) \), \( \nabla g(x) = \nabla f(x) \), and \( \nabla g - \nabla f \) is Lipschitz-continuous.

- Get \( O(1/k) \) and linear convergence rates depending on \( g - f \).
Big-N Problems

We want to minimize the sum of a finite set of smooth functions:

$$\min_{x \in \mathbb{R}^p} f(x) := \frac{1}{N} \sum_{i=1}^{N} f_i(x).$$
We want to minimize the sum of a finite set of smooth functions:

$$\min_{x \in \mathbb{R}^p} f(x) := \frac{1}{N} \sum_{i=1}^{N} f_i(x).$$

We are interested in cases where $N$ is very large.
We want to minimize the sum of a finite set of smooth functions:

$$\min_{x \in \mathbb{R}^p} f(x) := \frac{1}{N} \sum_{i=1}^{N} f_i(x).$$

We are interested in cases where $N$ is very large.

Simple example is least-squares,

$$f_i(x) := (a_i^T x - b_i)^2.$$

Other examples:
- logistic regression, Huber regression, smooth SVMs, CRFs, etc.
Stochastic vs. Deterministic Gradient Methods

- We consider minimizing \( f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \).
Stochastic vs. Deterministic Gradient Methods

- We consider minimizing \( f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \).
- Deterministic gradient method [Cauchy, 1847]:
  \[
  x_{t+1} = x_t - \alpha_t \nabla f(x_t) = x_t - \frac{\alpha_t}{N} \sum_{i=1}^{N} \nabla f_i(x_t).
  \]
  - Iteration cost is linear in \( N \).
  - Quasi-Newton methods still require \( O(N) \).
  - Convergence with constant \( \alpha_t \) or line-search.

Stochastic gradient method [Robbins & Monro, 1951]:
- Random selection of \( i(t) \) from \( \{1, 2, \ldots, N\} \).
- \( x_{t+1} = x_t - \alpha_t f'_i(x_t) \) gives unbiased estimate of true gradient,
  \[E[f'_i(x_t)] = \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x) = \nabla f(x)\]
- Iteration cost is independent of \( N \).
- As in subgradient method, we require \( \alpha_t \to 0 \).
- Classical choice is \( \alpha_t = O(1/t) \).
Stochastic vs. Deterministic Gradient Methods

- We consider minimizing \( f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \).
- **Deterministic** gradient method [Cauchy, 1847]:

\[
    x_{t+1} = x_t - \alpha_t \nabla f(x_t) = x_t - \frac{\alpha_t}{N} \sum_{i=1}^{N} \nabla f_i(x_t).
\]

- Iteration cost is **linear in** \( N \).
- Quasi-Newton methods still require \( O(N) \).
- Convergence with constant \( \alpha_t \) or line-search.

- **Stochastic** gradient method [Robbins & Monro, 1951]:
  - Random selection of \( i(t) \) from \( \{1, 2, \ldots, N\} \).
  
  \[
    x_{t+1} = x_t - \alpha_t f'_{i(t)}(x_t).
  \]
**Stochastic vs. Deterministic Gradient Methods**

- We consider minimizing $f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x)$.
- **Deterministic** gradient method [Cauchy, 1847]:

$$x_{t+1} = x_t - \alpha_t \nabla f(x_t) = x_t - \frac{\alpha_t}{N} \sum_{i=1}^{N} \nabla f_i(x_t).$$

- Iteration cost is **linear in $N$**.
- Quasi-Newton methods still require $O(N)$.
- Convergence with constant $\alpha_t$ or line-search.

- **Stochastic** gradient method [Robbins & Monro, 1951]:
  - Random selection of $i(t)$ from $\{1, 2, \ldots, N\}$.
  $$x_{t+1} = x_t - \alpha_t f'_i(t)(x_t).$$
  - Gives unbiased estimate of true gradient,

$$\mathbb{E}[f'_i(t)(x)] = \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x) = \nabla f(x).$$

- Iteration cost is **independent of $N$**.
Stochastic vs. Deterministic Gradient Methods

- We consider minimizing \( f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \).

- **Deterministic** gradient method [Cauchy, 1847]:

\[
x_{t+1} = x_t - \alpha_t \nabla f(x_t) = x_t - \frac{\alpha_t}{N} \sum_{i=1}^{N} \nabla f_i(x_t).
\]

  - Iteration cost is **linear in** \( N \).
  - Quasi-Newton methods still require \( O(N) \).
  - Convergence with constant \( \alpha_t \) or line-search.

- **Stochastic** gradient method [Robbins & Monro, 1951]:

  - Random selection of \( i(t) \) from \( \{1, 2, \ldots, N\} \).

\[
x_{t+1} = x_t - \alpha_t f'_{i(t)}(x_t).
\]

  - Gives unbiased estimate of true gradient,

\[
\mathbb{E}[f'_{i(t)}(x)] = \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x) = \nabla f(x).
\]

  - Iteration cost is **independent of** \( N \).
  - As in subgradient method, we require \( \alpha_t \to 0 \).
  - Classical choice is \( \alpha_t = O(1/t) \).
We consider minimizing $g(x) = \frac{1}{N} \sum_{i=1}^{n} f_i(x)$.

**Deterministic** gradient method [Cauchy, 1847]:

**Stochastic** gradient method [Robbins & Monro, 1951]:

Stochastic vs. Deterministic Gradient Methods
Convex Optimization Zoo

Stochastic iterations are $N$ times faster, but how many iterations?
## Convex Optimization Zoo

Stochastic iterations are $N$ times faster, but how many iterations?

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Exact</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly</td>
<td>$O(1/t)$</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

Good news for non-smooth problems: stochastic is as fast as deterministic.

We can solve non-smooth problems $N$ times faster!

Bad news for smooth problems: smoothness does not help stochastic methods.
Convex Optimization Zoo

Stochastic iterations are $N$ times faster, but how many iterations?

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Exact</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly</td>
<td>$O(1/t)$</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

- Good news for non-smooth problems:
  - stochastic is as fast as deterministic
- We can solve non-smooth problems $N$ times faster!
Convex Optimization Zoo

Stochastic iterations are $N$ times faster, but how many iterations?

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Exact</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subgradient</td>
<td>Convex</td>
<td>$O(1/\sqrt{t})$</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Subgradient</td>
<td>Strongly</td>
<td>$O(1/t)$</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>

- **Good news for non-smooth problems:**
  - stochastic is as fast as deterministic
- **We can solve non-smooth problems $N$ times faster!**
- **Bad news for smooth problems:**
  - smoothness does not help stochastic methods.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Assumptions</th>
<th>Exact</th>
<th>Stochastic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient</td>
<td>Convex</td>
<td>$O(1/t)$</td>
<td>$O(1/\sqrt{t})$</td>
</tr>
<tr>
<td>Gradient</td>
<td>Strongly</td>
<td>$O((1 - \mu/L)^t)$</td>
<td>$O(1/t)$</td>
</tr>
</tbody>
</table>
Deterministic vs. Stochastic Convergence Rates

Plot of convergence rates in smooth/strongly-convex case:

- **Stochastic**
- **Deterministic**

\[ \text{log(excess cost)} \]

\[ \text{time} \]
Speeding up Stochastic Gradient Methods

- We can try accelerated/Newton-like stochastic methods:
  - These do not improve on the convergence rates.
  - But may improve performance at start if noise is small.

\[ \alpha_t = O\left(\frac{1}{t^{\alpha}}\right) \text{ for } \alpha \in (0, 1) \text{ more robust than } O\left(\frac{1}{t}\right). \]

[Bach & Moulines, 2011]
Gradient averaging improves constants ('dual averaging').
[Nesterov, 2007]
Averaging later iterations gives rates without smoothness.
[Rakhlin et al., 2011]
Averaging in smooth case achieves same asymptotic rate as optimal stochastic Newton method.
[Polyak & Juditsky, 1992]
Constant step-size \( \alpha \) achieves rate of \( O\left(\rho t\right) + O\left(\alpha\right) \).
[Nedic & Bertsekas, 2000]
Averaging and constant step-size achieves \( O\left(\frac{1}{t}\right) \) rate for stochastic Newton-like methods without strong convexity.
[Bach & Moulines, 2013]
Speeding up Stochastic Gradient Methods

- We can try accelerated/Newton-like stochastic methods:
  - These do not improve on the convergence rates.
  - But may improve performance at start if noise is small.
- Ideas that do improve convergence: averaging and large steps:
Speeding up Stochastic Gradient Methods

- We can try accelerated/Newton-like stochastic methods:
  - These do not improve on the convergence rates.
  - But may improve performance at start if noise is small.

- Ideas that do improve convergence: **averaging** and **large steps**:
  - $\alpha_t = O(1/t^\alpha)$ for $\alpha \in (0.5, 1)$ more robust than $O(1/t)$.
    [Bach & Moulines, 2011]
We can try accelerated/Newton-like stochastic methods:
- These do not improve on the convergence rates.
- But may improve performance at start if noise is small.

Ideas that do improve convergence: averaging and large steps:
- $\alpha_t = O(1/t^\alpha)$ for $\alpha \in (0.5, 1)$ more robust than $O(1/t)$.
  [Bach & Moulines, 2011]
- Gradient averaging improves constants (‘dual averaging’).
  [Nesterov, 2007]
We can try accelerated/Newton-like stochastic methods:

- These do not improve on the convergence rates.
- But may improve performance at start if noise is small.

Ideas that do improve convergence: averaging and large steps:

- \( \alpha_t = O(1/t^\alpha) \) for \( \alpha \in (0.5, 1) \) more robust than \( O(1/t) \).
  - [Bach & Moulines, 2011]
- Gradient averaging improves constants (‘dual averaging’).
  - [Nesterov, 2007]
- Averaging later iterations gives rates without smoothness.
  - [Rakhlin et al., 2011]
**Speeding up Stochastic Gradient Methods**

- We can try accelerated/Newton-like stochastic methods:
  - These **do not** improve on the convergence rates.
  - But may improve performance at start if noise is small.

- Ideas that do improve convergence: **averaging** and **large steps**:
  - \( \alpha_t = O(1/t^\alpha) \) for \( \alpha \in (0.5, 1) \) more robust than \( O(1/t) \).
    - [Bach & Moulines, 2011]
  - Gradient averaging improves constants (‘dual averaging’).
    - [Nesterov, 2007]
  - Averaging later iterations gives rates without smoothness.
    - [Rakhlin et al., 2011]
  - Averaging in smooth case achieves **same asymptotic rate** as **optimal stochastic Newton** method. [Polyak & Juditsky, 1992]
Speeding up Stochastic Gradient Methods

- We can try accelerated/Newton-like stochastic methods:
  - These **do not** improve on the convergence rates.
  - But may improve performance at start if noise is small.

- Ideas that do improve convergence: **averaging** and **large steps**:
  - $\alpha_t = O(1/t^\alpha)$ for $\alpha \in (0.5, 1)$ more robust than $O(1/t)$.
    [Bach & Moulines, 2011]
  - Gradient averaging improves constants (‘dual averaging’).
    [Nesterov, 2007]
  - Averaging later iterations gives rates without smoothness.
    [Rakhlin et al., 2011]
  - Averaging in smooth case achieves **same asymptotic rate as optimal stochastic Newton** method.[Polyak & Juditsky, 1992]]
  - **Constant** step-size $\alpha$ achieves rate of $O(\rho^t) + O(\alpha)$.
    [Nedic & Bertsekas, 2000]
Speeding up Stochastic Gradient Methods

- We can try accelerated/Newton-like stochastic methods:
  - These do not improve on the convergence rates.
  - But may improve performance at start if noise is small.

- Ideas that do improve convergence: averaging and large steps:
  - $\alpha_t = O(1/t^\alpha)$ for $\alpha \in (0.5, 1)$ more robust than $O(1/t)$.
    [Bach & Moulines, 2011]
  - Gradient averaging improves constants (‘dual averaging’).
    [Nesterov, 2007]
  - Averaging later iterations gives rates without smoothness.
    [Rakhlin et al., 2011]
  - Averaging in smooth case achieves same asymptotic rate as optimal stochastic Newton method.[Polyak & Juditsky, 1992]]
  - Constant step-size $\alpha$ achieves rate of $O(\rho^t) + O(\alpha)$.
    [Nedic & Bertsekas, 2000]
  - Averaging and constant step-size achieves $O(1/t)$ rate for stochastic Newton-like methods without strong convexity.
    [Bach & Moulines, 2013]
Motivation for Hybrid Methods for Smooth Problems

- Stochastic vs. deterministic methods
  - Goal: best of both worlds
    - Linear rate with $O(1)$ iteration cost
    - $\log$ (excess cost)

**Graph:**
- Stochastic: Continuous curve
- Deterministic: Staircase approximation

**Axes:**
- Y-axis: $\log$ (excess cost)
- X-axis: Time
Motivation for Hybrid Methods for Smooth Problems

Stochastic vs. deterministic methods
• Goal = best of both worlds:
  - linear rate with $O(1)$ iteration cost
  - hybrid
  - $\log(\text{excess cost})$
  - stochastic
  - deterministic
  - time

\[
\text{log(excess cost)}
\]

\[
time
\]
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- **Can we have a rate of** $O(\rho^t)$ **with only 1 gradient evaluation per iteration?**
  - YES!
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?
  - YES! The **stochastic average gradient (SAG)** algorithm:
  - Randomly select $i(t)$ from $\{1, 2, \ldots, N\}$ and compute $f'_{i(t)}(x^t)$.

\[ x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} \nabla f_i(x^t) \]
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- Can we have a rate of $O(\rho^t)$ with only 1 gradient evaluation per iteration?
  - YES! The stochastic average gradient (SAG) algorithm:
    - Randomly select $i(t)$ from \{1, 2, \ldots, N\} and compute $f'_{i(t)}(x^t)$.
    
    $$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} \nabla f_i(x^t)$$

[Blatt et al., 2007] Assumes gradients of non-selected examples don't change.
Assumption becomes accurate as $||x^{t+1} - x^t|| \rightarrow 0$.
Memory requirements reduced to $O(N)$ for many problems.
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- **Can we have a rate of** $O(\rho^t)$ **with only 1 gradient evaluation per iteration?**
  - YES! The **stochastic average gradient (SAG) algorithm:**
    - Randomly select $i(t)$ from $\{1, 2, \ldots, N\}$ and compute $f'_i(x^t)$.
    
    $$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} y^t_i$$

- **Memory:** $y^t_i = \nabla f_i(x^t)$ from the last $t$ where $i$ was selected.
  
  [Le Roux et al., 2012]
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- **Can we have a rate of** $O(\rho^t)$ **with only 1 gradient evaluation per iteration?**
  - YES! The **stochastic average gradient (SAG)** algorithm:
    - Randomly select $i(t)$ from $\{1, 2, \ldots, N\}$ and compute $f_{i(t)}'(x^t)$.
    
    $$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} y_i^t$$

    - **Memory**: $y_i^t = \nabla f_i(x^t)$ from the last $t$ where $i$ was selected.
      [Le Roux et al., 2012]

- **Stochastic** variant of increment average gradient (IAG).
  [Blatt et al., 2007]
Stochastic Average Gradient

- Should we use stochastic methods for smooth problems?
- **Can we have a rate of** $O(\rho^t)$ **with only 1 gradient evaluation per iteration?**
  - YES! The **stochastic average gradient (SAG)** algorithm:
    - Randomly select $i(t)$ from \{1, 2, …, $N$\} and compute $f'_{i(t)}(x^t)$.
    
    $$x^{t+1} = x^t - \frac{\alpha^t}{N} \sum_{i=1}^{N} y_i^t$$

  - **Memory**: $y_i^t = \nabla f_i(x^t)$ from the last $t$ where $i$ was selected. [Le Roux et al., 2012]

- **Stochastic** variant of increment average gradient (IAG).
  - [Blatt et al., 2007]

- Assumes gradients of non-selected examples don’t change.
- Assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.
- Memory requirements reduced to $O(N)$ for many problems.
### Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Rate</th>
<th>Grads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic Gradient</td>
<td>$O(1/t)$</td>
<td>1</td>
</tr>
<tr>
<td>Gradient</td>
<td>$O((1 - \mu/L)^t)$</td>
<td>N</td>
</tr>
</tbody>
</table>

$L_i$ is the Lipschitz constant over all $\nabla f_i$. SAG has a similar speed to the gradient method, but only looks at one training example per iteration. Recent work extends this result in various ways: similar rates for stochastic dual coordinate ascent. [Shalev-Schwartz & Zhang, 2013] Memory-free variants. [Johnson & Zhang, 2013, Madavi et al., 2013] Proximal-gradient variants. [Mairal, 2013] ADMM variants. [Wong et al., 2013] Improved constants. [Defazio et al., 2014] Non-uniform sampling. [Schmidt et al., 2013]
## Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Rate</th>
<th>Grads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic Gradient</td>
<td>$O(1/t)$</td>
<td>1</td>
</tr>
<tr>
<td>Gradient</td>
<td>$O((1 - \mu/L)^t)$</td>
<td>N</td>
</tr>
<tr>
<td>SAG</td>
<td>$O((1 - \min{\frac{\mu}{16L}, \frac{1}{8N}})^t)$</td>
<td>1</td>
</tr>
</tbody>
</table>

$L_i$ is the Lipschitz constant over all $\nabla f_i$ ($L_i \geq L$).

SAG has a similar speed to the gradient method, but only looks at one training example per iteration.

Recent work extends this result in various ways:
- Similar rates for stochastic dual coordinate ascent. [Shalev-Schwartz & Zhang, 2013]
- Memory-free variants. [Johnson & Zhang, 2013, Madavi et al., 2013]
- Proximal-gradient variants. [Mairal, 2013]
- ADMM variants. [Wong et al., 2013]
- Improved constants. [Defazio et al., 2014]
- Non-uniform sampling. [Schmidt et al., 2013]
Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Rate</th>
<th>Grads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic</td>
<td>$O(1/t)$</td>
<td>1</td>
</tr>
<tr>
<td>Gradient</td>
<td>$O((1 - \mu/L)^t)$</td>
<td>N</td>
</tr>
<tr>
<td>SAG</td>
<td>$O((1 - \min{\frac{\mu}{16L_i}, \frac{1}{8N}})^t)$</td>
<td>1</td>
</tr>
</tbody>
</table>

- $L_i$ is the Lipschitz constant over all $\nabla f_i$ ($L_i \geq L$).
- SAG has a similar speed to the gradient method, but only looks at one training example per iteration.

Recent work extends this result in various ways:
- Similar rates for stochastic dual coordinate ascent. [Shalev-Schwartz & Zhang, 2013]
- Memory-free variants. [Johnson & Zhang, 2013, Madavi et al., 2013]
- Proximal-gradient variants. [Mairal, 2013]
- ADMM variants. [Wong et al., 2013]
- Improved constants. [Defazio et al., 2014]
- Non-uniform sampling. [Schmidt et al., 2013]
Convex Optimization Zoo

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Rate</th>
<th>Grads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic Gradient</td>
<td>$O(1/t)$</td>
<td>1</td>
</tr>
<tr>
<td>Gradient</td>
<td>$O((1 - \mu/L)^t)$</td>
<td>N</td>
</tr>
<tr>
<td>SAG</td>
<td>$O((1 - \min{\mu/16L_i, \frac{1}{8N}})^t)$</td>
<td>1</td>
</tr>
</tbody>
</table>

- $L_i$ is the Lipschitz constant over all $\nabla f_i$ ($L_i \geq L$).
- **SAG** has a similar speed to the gradient method, but only looks at one training example per iteration.
- Recent work extends this result in various ways:
  - Similar rates for stochastic dual coordinate ascent. [Shalev-Schwartz & Zhang, 2013]
  - Memory-free variants. [Johnson & Zhang, 2013, Madavi et al., 2013]
  - Proximal-gradient variants. [Mairal, 2013]
  - ADMM variants. [Wong et al., 2013]
  - Improved constants. [Defazio et al., 2014]
  - Non-uniform sampling. [Schmidt et al., 2013]
Comparing FG and SG Methods

- quantum \((n = 50000, p = 78)\) and rcv1 \((n = 697641, p = 47236)\)
SAG Compared to FG and SG Methods

- quantum \((n = 50000, p = 78)\) and rcv1 \((n = 697641, p = 47236)\)
Consider problems of the form

\[
\min_x f(Ax) + \sum_{i=1}^n h_i(x_i), \quad \min_x \sum_{i \in \mathcal{V}} f_i(x_i) + \sum_{(i,j) \in \mathcal{E}} f_{ij}(x_i, x_j),
\]

where \( f \) and \( f_{ij} \) are smooth and \( G = \{ \mathcal{V}, \mathcal{E} \} \) is a graph.
Coordinate Descent Methods

Consider problems of the form

\[
\min_x f(Ax) + \sum_{i=1}^n h_i(x_i), \quad \min_x \sum_{i \in \mathcal{V}} f_i(x_i) + \sum_{(i,j) \in \mathcal{E}} f_{ij}(x_i, x_j),
\]

where \( f \) and \( f_{ij} \) are smooth and \( G = \{ \mathcal{V}, \mathcal{E} \} \) is a graph.

Appealing strategy for these problems is coordinate descent:

\[
x_j^+ = x_j - \alpha \nabla f(x).
\]

(i.e., update one variable at a time)

We can typically perform a cheap and precise line-search for \( \alpha \).
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg\max_j \{|\nabla_j f(x)|\}$.
  
  (but only efficient to calculate in some special cases)
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg\max_j \{ |\nabla_j f(x)| \}$. (but only efficient to calculate in some special cases)
- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  \[ O((1 - \mu/L_j D)^t) \]
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg \max_j \{|\nabla_j f(x)|\}$.
  (but only efficient to calculate in some special cases)
- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  $$O((1 - \mu/L_j D)^t)$$
- $L_j$ is typically much smaller than $L$ across all coordinates:
Convex Functions Smooth Optimization Non-Smooth Optimization Randomized Algorithms Parallel/Distributed Optimization

Convergence Rate of Coordinate Descent

- The **steepest descent** choice is $j = \arg \max_j \{ |\nabla_j f(x)| \}$.  
  (but only efficient to calculate in some special cases)
- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  $$O((1 - \mu/L_j D)^t)$$
- $L_j$ is typically much smaller than $L$ across all coordinates:
  - Coordinate descent is faster if we can do $D$ coordinate descent steps for cost of one gradient step.

[Choosing a random coordinate $j$ has same rate as steepest coordinate descent.][Nesterov, 2010]

Various extensions:
- Non-uniform sampling (Lipschitz sampling) [Nesterov, 2010]
- Projected coordinate descent (product constraints) [Nesterov, 2010]
- Proximal coordinate descent (separable non-smooth term) [Richtarik & Takac, 2011]
- Frank-Wolfe coordinate descent (product constraints) [LaCoste-Julien et al., 2013]
- Accelerated version [Fercoq & Richtarik, 2013] (exact step size for $\ell_1$-regularized least squares)
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg \max_j \{|\nabla_j f(x)|\}$.
  (but only efficient to calculate in some special cases)

- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  \[O((1 - \mu/L_j D)^t)\]

- $L_j$ is typically much smaller than $L$ across all coordinates:
  - Coordinate descent is faster if we can do $D$ coordinate descent steps for cost of one gradient step.

- Choosing a *random coordinate* $j$ has same rate as steepest coordinate descent.[Nesterov, 2010]
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg \max_j \{|\nabla_j f(x)|\}$. (but only efficient to calculate in some special cases)
- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  $$O((1 - \mu/L_j D)^t)$$
- $L_j$ is typically much smaller than $L$ across all coordinates:
  - Coordinate descent is faster if we can do $D$ coordinate descent steps for cost of one gradient step.
- Choosing a random coordinate $j$ has same rate as steepest coordinate descent. [Nesterov, 2010]
- Various extensions:
  - Non-uniform sampling (Lipschitz sampling) [Nesterov, 2010]
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg \max_j \{|\nabla_j f(x)|\}$.
  
  (but only efficient to calculate in some special cases)

- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  
  $$O((1 - \mu/L_j D)^t)$$

- $L_j$ is typically much smaller than $L$ across all coordinates:
  
  Coordinate descent is faster if we can do $D$ coordinate descent steps for cost of one gradient step.

- Choosing a random coordinate $j$ has same rate as steepest coordinate descent. [Nesterov, 2010]

- Various extensions:
  
  - Non-uniform sampling (Lipschitz sampling) [Nesterov, 2010]
  - Projected coordinate descent (product constraints) [Nesterov, 2010]
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is \( j = \arg \max_j \{|\nabla_j f(x)|\} \).
  
  (but only efficient to calculate in some special cases)

- Convergence rate (strongly-convex, partials are \( L_j \)-Lipschitz):
  \[
  O((1 - \mu/L_j D)^t)
  \]

- \( L_j \) is typically much smaller than \( L \) across all coordinates:
  - Coordinate descent is faster if we can do \( D \) coordinate descent steps for cost of one gradient step.

- Choosing a random coordinate \( j \) has same rate as steepest coordinate descent.[Nesterov, 2010]

- Various extensions:
  - Non-uniform sampling (Lipschitz sampling) [Nesterov, 2010]
  - Projected coordinate descent (product constraints) [Nesterov, 2010]
  - Proximal coordinate descent (separable non-smooth term) [Richtarik & Takac, 2011]
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg \max_j \{|\nabla_j f(x)|\}$.
  
  (but only efficient to calculate in some special cases)

- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  \[ O((1 - \mu/L_j D)^t) \]

- $L_j$ is typically much smaller than $L$ across all coordinates:
  - Coordinate descent is faster if we can do $D$ coordinate descent steps for cost of one gradient step.

- Choosing a random coordinate $j$ has same rate as steepest coordinate descent.[Nesterov, 2010]

- Various extensions:
  - Non-uniform sampling (Lipschitz sampling) [Nesterov, 2010]
  - Projected coordinate descent (product constraints) [Nesterov, 2010]
  - Proximal coordinate descent (separable non-smooth term) [Richtarik & Takac, 2011]
  - Frank-Wolfe coordinate descent (product constraints) [LaCoste-Julien et al., 2013]
Convergence Rate of Coordinate Descent

- The *steepest descent* choice is $j = \arg \max_j \{ |\nabla_j f(x)| \}$.
  
  (but only efficient to calculate in some special cases)

- Convergence rate (strongly-convex, partials are $L_j$-Lipschitz):
  
  $$O((1 - \mu / L_j D)^t)$$

- $L_j$ is typically much smaller than $L$ across all coordinates:
  - Coordinate descent is faster if we can do $D$ coordinate descent steps for cost of one gradient step.

- Choosing a random coordinate $j$ has same rate as steepest coordinate descent.[Nesterov, 2010]

- Various extensions:
  - Non-uniform sampling (Lipschitz sampling) [Nesterov, 2010]
  - Projected coordinate descent (product constraints) [Nesterov, 2010]
  - Proximal coordinate descent (separable non-smooth term) [Richtarik & Takac, 2011]
  - Frank-Wolfe coordinate descent (product constraints) [LaCoste-Julien et al., 2013]
  - Accelerated version [Fercoq & Richtarik, 2013]
Randomized Linear Algebra

Consider problems of the form

$$\min_x f(Ax),$$

where bottleneck is matrix multiplication and $A$ is low-rank.

"Halko et al., 2011, Mahoney, 2011"

May work quite badly if singular values decay slowly.
Randomized Linear Algebra

- Consider problems of the form

\[
\min_x f(Ax),
\]

where bottleneck is matrix multiplication and \( A \) is low-rank.

- Randomized linear algebra approaches uses

\[
A \approx Q(Q^T A),
\]

or choses random row/columns subsets.
Consider problems of the form

$$\min_x f(Ax),$$

where bottleneck is matrix multiplication and $A$ is low-rank.

Randomized linear algebra approaches uses

$$A \approx Q(Q^TA),$$

or choses random row/columns subsets.

Now work with $f(Q(Q^TA))$. 

[Halko et al., 2011, Mahoney, 2011]
Consider problems of the form
\[
\min_x f(Ax),
\]
where bottleneck is matrix multiplication and \( A \) is low-rank.

Randomized linear algebra approaches use
\[
A \approx Q(Q^T A),
\]
or choses random row/columns subsets.

Now work with \( f(Q(Q^T A)) \).

\( Q \) formed from Gram-Schmidt and matrix multiplication with random vectors gives very good approximation bounds, if singular values decay quickly.[Halko et al., 2011, Mahoney, 2011]
Randomized Linear Algebra

- Consider problems of the form
  \[ \min_x f(Ax), \]
  where bottleneck is matrix multiplication and \( A \) is low-rank.
- Randomized linear algebra approaches uses
  \[ A \approx Q(Q^T A), \]
  or choses random row/columns subsets.
- Now work with \( f(Q(Q^T A)) \).
- \( Q \) formed from Gram-Schmidt and matrix multiplication with random vectors gives very good approximation bounds, if singular values decay quickly.[Halko et al., 2011, Mahoney, 2011]
- May work quite badly if singular values decay slowly.
Outline

1. Convex Functions
2. Smooth Optimization
3. Non-Smooth Optimization
4. Randomized Algorithms
5. Parallel/Distributed Optimization
Motivation for Parallel and Distributed

- Two recent trends:
  - We aren't making large gains in serial computation speed.
  - Datasets no longer fit on a single machine.
Motivation for Parallel and Distributed

- Two recent trends:
  - We aren’t making large gains in serial computation speed.
  - Datasets no longer fit on a single machine.

- Result: we must use parallel and distributed computation.

- Two major issues:
  - Synchronization: we can’t wait for the slowest machine.
  - Communication: we can’t transfer all information.
A lot of machine learning problems are **embarrassingly parallel**:

- Split task across $M$ machines, solve independently, combine.
A lot of machine learning problems are **embarrassingly parallel**:

- Split task across $M$ machines, solve independently, combine.

E.g., computing the gradient in deterministic gradient method,

$$
\frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x) = \frac{1}{N} \left( \sum_{i=1}^{N/M} \nabla f_i(x) + \sum_{i=(N/M)+1}^{2N/M} \nabla f_i(x) + \ldots \right).
$$
Embarassing Parallelism in Machine Learning

- A lot of machine learning problems are embarrassingly parallel:
  - Split task across $M$ machines, solve independently, combine.

- E.g., computing the gradient in deterministic gradient method,

\[
\frac{1}{N} \sum_{i=1}^{N} \nabla f_i(x) = \frac{1}{N} \left( \sum_{i=1}^{N/M} \nabla f_i(x) + \sum_{i=(N/M)+1}^{2N/M} \nabla f_i(x) + \ldots \right).
\]

- These allow optimal linear speedups.
  - You should always consider this first!
Asynchronous Computation

- Do we have to wait for the last computer to finish?

\[
x_{k+1} = x_k - \alpha \nabla f_{i_k}(x_k - m_k)
\]

You need to decrease step-size in proportion to asynchrony. Convergence rate decays elegantly with delay \(m\). [Niu et al., 2011]
Asynchronous Computation

- Do we have to wait for the last computer to finish?
  - No!
- Updating asynchronously saves a lot of time.
Asynchronous Computation

- Do we have to wait for the last computer to finish?
- No!
- Updating asynchronously saves a lot of time.
- E.g., stochastic gradient method on shared memory:

$$x^{k+1} = x^k - \alpha \nabla f_{i_k}(x^{k-m}).$$
Asynchronous Computation

- Do we have to wait for the last computer to finish?
- No!
- Updating asynchronously saves a lot of time.
- E.g., stochastic gradient method on shared memory:

  \[ x^{k+1} = x^k - \alpha \nabla f_{i_k}(x^k - m). \]

- You need to decrease step-size in proportion to asynchrony.
- Convergence rate decays elegantly with delay \( m \). [Niu et al., 2011]
Reduced Communication: Parallel Coordinate Descent

- It may be expensive to communicate parameters $x$. 

\[
\begin{align*}
  x_j^1 &= x_j^1 - \alpha_j^1 \nabla_j^1 f(x) \\
  x_j^2 &= x_j^2 - \alpha_j^2 \nabla_j^2 f(x) \\
  x_j^3 &= x_j^3 - \alpha_j^3 \nabla_j^3 f(x)
\end{align*}
\]

Speedup is based on density of graph. [Richtarik & Takac, 2013]
Reduced Communication: Parallel Coordinate Descent

- It may be expensive to communicate parameters $x$.
- One solution: use parallel coordinate descent:

$$x_{j_1} = x_{j_1} - \alpha_{j_1} \nabla_{j_1} f(x)$$
$$x_{j_2} = x_{j_2} - \alpha_{j_2} \nabla_{j_2} f(x)$$
$$x_{j_3} = x_{j_3} - \alpha_{j_3} \nabla_{j_3} f(x)$$

- Only needs to communicate single coordinates.
Reduced Communication: Parallel Coordinate Descent

- It may be expensive to communicate parameters $x$.
- One solution: use parallel coordinate descent:

  $$x_{j1} = x_{j1} - \alpha_{j1} \nabla_{j1} f(x)$$
  $$x_{j2} = x_{j2} - \alpha_{j2} \nabla_{j2} f(x)$$
  $$x_{j3} = x_{j3} - \alpha_{j3} \nabla_{j3} f(x)$$

- Only needs to communicate single coordinates.
- Again need to decrease step-size for convergence.
- Speedup is based on density of graph.[Richtarik & Takac, 2013]
Reduced Communication: Decentralized Gradient

- We may need to distribute the data across machines.
- We may not want to update a ‘centralized’ vector $\mathbf{x}$. 

\[ \mathbf{x}_c = \frac{1}{|\text{nei}(c)|} \sum_{c' \in \text{nei}(c)} \mathbf{x}_{c'} - \alpha \mathbf{M} \sum_{i=1}^{m} \nabla f_i(\mathbf{x}_{c}) \]

Gradient descent is a special case where all neighbours communicate. With modified update, rate decays gracefully as the graph becomes sparse.

[Shi et al., 2014]

Can also consider communication failures.

[Agarwal & Duchi, 2011]
Reduced Communication: Decentralized Gradient

- We may need to distribute the data across machines.
- We may not want to update a ‘centralized’ vector $x$.
- One solution: **decentralized gradient method**:
  - Each processor has its own data samples $f_1, f_2, \ldots f_m$.
  - Each processor has its own parameter vector $x_c$.
  - Each processor only communicates with a limited number of neighbours $\text{nei}(c)$.

Gradient descent is a special case where all neighbours communicate. With modified update, rate decays gracefully as the graph becomes sparse.

[Shi et al., 2014]
[Agarwal & Duchi, 2011]
Reduced Communication: Decentralized Gradient

- We may need to distribute the data across machines.
- We may not want to update a ‘centralized’ vector $x$.
- One solution: decentralized gradient method:
  - Each processor has its own data samples $f_1, f_2, \ldots f_m$.
  - Each processor has its own parameter vector $x_c$.
  - Each processor only communicates with a limited number of neighbours $\text{nei}(c)$.

$$x_c = \frac{1}{|\text{nei}(c)|} \sum_{c' \in \text{nei}(c)} x_{c'} - \frac{\alpha_c}{M} \sum_{i=1}^{M} \nabla f_i(x_c).$$

- Gradient descent is special case where all neighbours communicate.
- With modified update, rate decays gracefully as graph becomes sparse. [Shi et al., 2014]
- Can also consider communication failures. [Agarwal & Duchi, 2011]
Summary:

- **Part 1**: Convex functions have special properties that allow us to efficiently minimize them.

- **Part 2**: Gradient-based methods allow elegant scaling with problems sizes for smooth problems.

- **Part 3**: Tricks like proximal-gradient methods allow the same scaling for many non-smooth problems.

- **Part 4**: Randomized algorithms allow even further scaling for problem structures that commonly arise in machine learning.

- **Part 5**: The future will require parallel and distributed that are asynchronous and are careful about communication costs.

Thank you for coming and staying until the end!
For non-convex function we have
\[ \|\nabla f(x^j)\|^2 = O(1/t), \]
for at least one \( j \) in the sequence. [Ghadimi & Lan, 2013]

Recall the rate for non-convex optimization with grid-search,
\[ f(x) - f(x^*) = O(1/t^{1/d}). \]

Bayesian optimization can improve the dependence on \( d \) if the function is sufficiently smooth. [Bull, 2011]