CPSC 540: Machine Learning

Directed Acyclic Graphical Models

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Last Time: Viterbi Decoding and Message Passing

- Decoding in density models: finding x with highest joint probability:

$$\underset{x_1,x_2,\dots,x_d}{\operatorname{argmax}}\ p(x_1,x_2,\dots,x_d).$$

- For Markov chains, we find decoding by writing maximization as

$$\max_{x_1,x_2,x_3,x_4} p(x_1,x_2,x_3,x_4) = \max_{x_4} \underbrace{\max_{x_3} p(x_4 \mid x_3) \underbrace{\max_{x_2} p(x_3 \mid x_2) \underbrace{\max_{x_1} p(x_2 \mid x_1) \underbrace{p(x_1}_{M_1(x_1)})}_{M_2(x_2)}}_{M_3(x_3)}}_{M_4(x_4)},$$

- Viterbi decoding computes $M_1(x_1)$ for all x_1, $M_2(x_2)$ for all x_2, and so on.
  The $M_j(x_j)$ functions are called messages (summarize everything about past).

Chapman-Kolmogorov Equations as Message Passing

- We can also view Chapman Kolmogorov equations as message passing:

$$
\begin{aligned}
\sum_{x_1}\sum_{x_2}\sum_{x_3}\sum_{x_4} p(x_1, x_2, x_3, x_4) &= \sum_{x_4}\sum_{x_3}\sum_{x_2}\sum_{x_1} p(x_4 \mid x_3)p(x_3 \mid x_2)p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x4}\sum_{x3} p(x_4 \mid x_3)\sum_{x_2} p(x_3 \mid x_2)\sum_{x_1} p(x_2 \mid x_1)M_1(x_1) \\
&= \sum_{x4}\sum_{x3} p(x_4 \mid x_3)\sum_{x_2} p(x_3 \mid x_2)M_2(x_2) \\
&= \sum_{x4}\sum_{x3} p(x_4 \mid x_3)M_3(x_3) \\
&= \sum_{x4} M_4(x_4),
\end{aligned}
$$

- Messages $M_j(x_j)$ are the marginals of the Markov chain.
  - So we can view CK equations as Viterbi decoding with "max" replace by "sum".
  - Also known as "max-product" and "sum-product" algorithms.

Message-Passing Algorithms

- We've discussed several algorithms with similar structure:
  - Viterbi decoding algorithm for decoding in discrete Markov chains.
  - CK equations for marginals in discrete Markov chains.
  - Gaussian updates for marginals in Gaussian Markov chains.

- These are all special cases of message-passing algorithms:
  1. Define M_j summarizing all relevant information about the past at time j.
  2. Use Markov property to write M_j recursively in terms of M_{j-1}.
  3. Solve task by computing $M_1, M_2, \ldots, M_d$.

- "Generalized distributive law" is a framework for describing when/why this works:
  - `https://authors.library.caltech.edu/1541/1/AJIieeetit00.pdf`

- In some cases we'll also need backwards message V_j ("cost to go"):
  - V_j summarizes all relevant information about the future at time j.

Conditionals via Backwards Messages

- Markov chain decoding using backwards messages $V_j(x_j)$:

$$\begin{aligned}
\max_{x_1}\max_{x_2}\max_{x_3}\max_{x_4} p(x_1,x_2,x_3,x_4) &= \max_{x_1}\max_{x_2}\max_{x3}\max_{x4} p(x_4 \mid x_3)p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3) \\
&= \max_{x_1} p(x_1)\max_{x_2} p(x_2 \mid x_1)\max_{x3} p(x_3 \mid x_2)\max_{x4} p(x_4 \mid x_3) \\
&= \max_{x_1} p(x_1)\max_{x_2} p(x_2 \mid x_1)\max_{x3} p(x_3 \mid x_2)\max_{x4} p(x_4 \mid x_3)\underbrace{V_4(x_4)}_{=1} \\
&= \max_{x_1} p(x_1)\max_{x_2} p(x_2 \mid x_1)\max_{x3} p(x_3 \mid x_2)V_3(x_3) \\
&= \max_{x_1=c} p(x_1)\max_{x_2} p(x_2 \mid x_1)V_2(x_2) \\
&= \max_{x_1} p(x_1)V_1(x_1).
\end{aligned}$$

- Computing all $M_j(x_j)$ and $V_j(x_j)$ is called forward backward algorithm.
  - Important later to compute marginals in generalizations of Markov chains.
  - Can be used to efficiently compute conditionals (bonus).

Outline

Higher-Order Markov Models

- Markov models use a density of the form

$$p(x) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)p(x_4 \mid x_3) \cdots p(x_d \mid x_{d-1}).$$

- They support efficient computation but Markov assumption is strong.

- A more flexible model would be a second-order Markov model,

$$p(x) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2, x_1)p(x_4 \mid x_3, x_2) \cdots p(x_d \mid x_{d-1}, x_{d-2}),$$

  or even a higher-order models.

- General case is called directed acyclic graphical (DAG) models:
  - They allow dependence on any subset of previous features.

DAG Models

- As in Markov chains, DAG models use the chain rule to write

$$p(x_1, x_2, \dots, x_d) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \cdots p(x_d \mid x_1, x_2, \dots, x_{d-1}).$$

- We can alternately write this as:

$$p(x_1, x_2, \dots, x_d) = \prod_{j=1}^{d} p(x_j \mid x_{1:j-1}).$$

- In Markov chains, we assumed x_j only depends on previous x_{j-1} given past.
- In DAGs, x_j can depend on any subset of the past $x_1, x_2, \dots, x_{j-1}$.

DAG Models

- To reduce number of parameters, in DAG models we use

$$p(x_1, x_2, \ldots, x_d) = \prod_{j=1}^{d} p(x_j \mid x_{\text{pa}(j)}),$$

  where $\text{pa}(j)$ are the "parents" of node j.
  - For Markov chains the only "parent" of j is $(j-1)$.
  - If we have k parents we only need 2^{k+1} parameters.

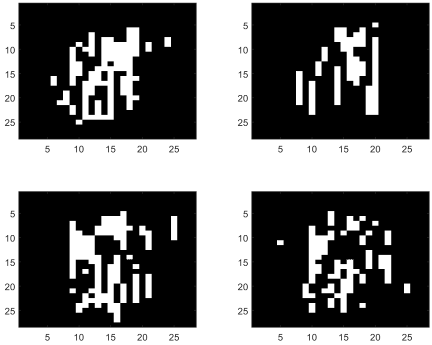
- This corresponds to a set of conditional independence assumptions,

$$p(x_j \mid x_{1:j-1}) = p(x_j \mid x_{\text{pa}(j)}),$$

  that we're independent of previous non-parents given the parents.

MNIST DIgits with Markov Chains

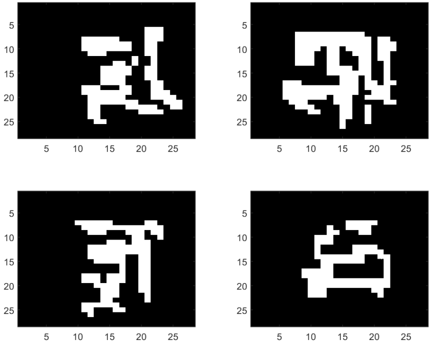
- Recall trying to model digits using an inhomogeneous Markov chain:



Only models dependence on pixel above, not on 2 pixels above nor across columns.

MNIST Digits with DAG Model (Sparse Parents)

- Samples from a DAG model with 8 parents per feature:



Parents of (i,j) are 8 other pixels in the neighbourhood $(i-2:i, j-2:j)$:

$$\{(i{-}2,j{-}2),(i{-}1,j{-}2),(i,j{-}2),(i{-}2,j{-}1),(i{-}1,j{-}1),(i,j{-}1),(i{-}2,j),(i{-}1,j)\}.$$

From Probability Factorizations to Graphs

- DAG models are also known as "Bayesian networks" and "belief networks".
- "Graphical" name comes from visualizing features/parents as a graph:
  - We have a node for each variable j.
  - We place an edge into j from each of its parents.
- The DAG representation for a Markov chains is:

  - Different than "state transition diagrams": edges are between variables (not states).
- This graph is not just a visualization tool:
  - Can be used to test arbitrary conditional independences ("d-separation").
  - Graph structure tells us whether message passing is efficient ("treewidth").

Graph Structure Examples

With product of independent we have

$$p(x) = \prod_{j=1}^{d} p(x_j),$$

so $\text{pa}(j) = \varnothing$ and the graph is:

Graph Structure Examples

With Markov chain we have

$$p(x) = p(x_1) \prod_{j=2}^{d} p(x_j \mid x_{j-1}),$$

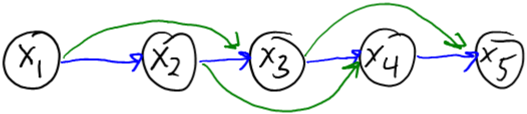
so $\text{pa}(j) = \{j-1\}$ and the graph is:

Graph Structure Examples

With second-order Markov chain we have

$$p(x) = p(x_1)p(x_2 \mid x_1) \prod_{j=3}^{d} p(x_j \mid x_{j-1}, x_{j-2}),$$

so $\mathrm{pa}(j) = \{j-2, j-1\}$ and the graph is:

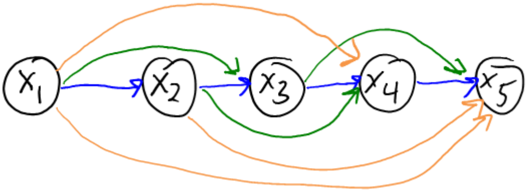


Graph Structure Examples

With general distribution we have

$$p(x) = \prod_{j=1}^{d} p(x_j \mid x_{1:j-1}).$$

so $\mathrm{pa}(j) = \{1, 2, \ldots, j-1\}$ and the graph is:

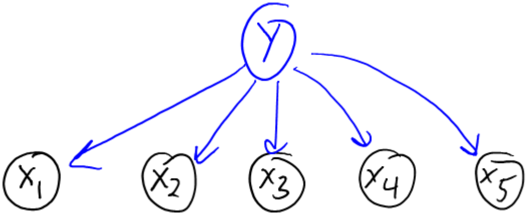


Graph Structure Examples

In naive Bayes we add an extra variable y and use

$$p(y, x) = p(y) \prod_{j=1}^{d} p(x_j \mid y),$$

which has $\text{pa}(y) = \emptyset$ and $\text{pa}(x_j) = y$ giving

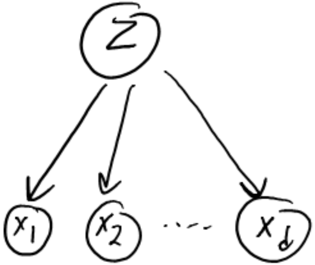


Graph Structure Examples

With mixture of independent models we have

$$p(z, x) = p(z) \prod_{j=1}^{d} p(x_j \mid z).$$

which has pa$(z) = \emptyset$ and pa$(x_j) = z$ giving same structured as naive Bayes:



Graph Structure Examples

- Instead of factorizing by variables j, could factor into blocks b:

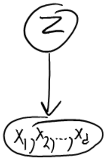
$$p(x) = \prod_b p(x_b \mid x_{\text{pa}(b)}),$$

  and have the nodes be blocks (we assume full connectivity within the block).

- With mixture of Gaussian and full covariances we have

$$p(z, x) = p(z)p(x \mid z).$$

- The corresponding graph structure is:



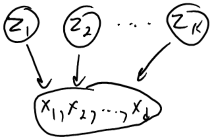
- Gaussian generative classifiers (GDA) have the same structure.
  - But using class lable y instead of cluster z.

Graph Structure Examples

With probabilistic PCA we have

$$p(z, x) = p(x \mid z) \prod_{c=1}^{k} p(z_c).$$

The corresponding graph structure is:

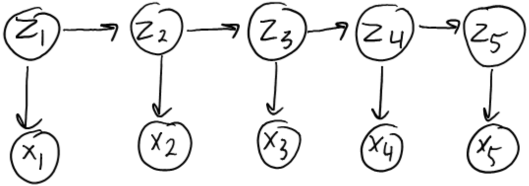


The data x comes from a set of independent parents (latent factors).

Graph Structure Examples

Sometimes it's easier to present a model using the graph.

Later in the course we'll see hidden Markov models which have this structure:

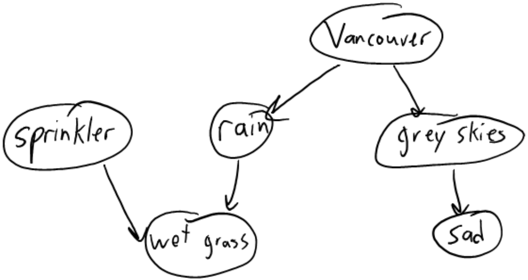


You should already be able to get an idea of what this model does:

- We have hidden variables z_j that follow a Markov chain.
- Each feature x_j depends on corresponding hidden variable z_j.

Graph Structure Examples

We can consider less-structured examples,

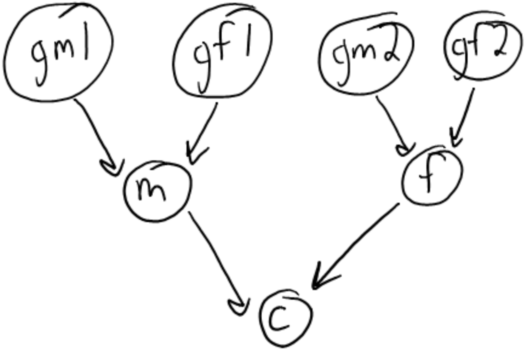


The corresponding factorization is:

$$p(S, V, R, W, G, D) = p(S)p(V)p(R \mid V)p(W \mid S, R)p(G \mid V)p(D \mid G).$$

Graph Structure Examples

We can consider phylogeny (family trees):



Outline

DAGs and Conditional Independence

- In DAGs we make the conditional independence assumption that

$$p(x_j \mid x_{j-1}, x_{j-2}, \dots, x_1) = p(x_j \mid x_{\text{pa}}(j)).$$

- But these conditional independence assumptions can imply other assumptions.
  - For example, in Markov chains we directly assume for all j that

$$p(x_j \mid x_{j-1}, x_{j-2}, \dots, x_1) = p(x_j \mid x_{j-1}),$$

    but this also implies that

$$p(x_j \mid x_{j-2}, x_{j-3}, \dots, x_1) = p(x_j \mid x_{j-2}),$$

    and it implies that

$$p(x_j \mid x_{j+1}, x_{j+2}, \dots, x_d) = p(x_j \mid x_{j+1}).$$

- Knowing which assumptions hold can help identify which operations are efficient.
  - For example, decoding in generals DAGs is NP-hard but it's easy in Markov chains.

Review of Independence

- Let A and B are random variables taking values $a \in \mathcal{A}$ and $b \in \mathcal{B}$.
- We say that A and B are independent if we have

  $$p(a, b) = p(a)p(b),$$

  for all a and b.
- To denote independence of x_i and x_j we use the notation

  $$x_i \perp x_j.$$

Review of Independence

- For independent a and b we have

$$p(a \mid b) = \frac{p(a,b)}{p(b)} = \frac{p(a)p(b)}{p(b)} = p(a).$$

- This gives us a more intuitive definition: A and B are independent if

$$p(a \mid b) = p(a)$$

  for all a and $b \neq 0$.
  - In words: knowing b tells us nothing about a (and vice versa).

- Useful fact: $a \perp b$ iff $p(a,b) = f(a)g(b)$ for some functions f and g.

Example: Independence in Product Models

- Let's show independence of pairs x_i and x_j in product of independent models:

$$p(x_1, x_2, \ldots, x_d) = p(x_1)p(x_2)\cdots p(x_d).$$

- From marginalization rule we have

$$p(x_i, x_j) = \sum_{x_{-ij}} p(x_1, x_2, \ldots, x_d),$$

where x_{-ij} is "over all variables except i and j".

- Using the definition of $p(x)$ above we get

$$p(x_i, x_j) = \sum_{x_{-ij}} p(x_1)p(x_2)\cdots p(x_d) = p(x_i)p(x_j) \underbrace{\sum_{x_{-ij}} \prod_{j' \neq i, j' \neq j} p(x_{j'})}_{=1} = p(x_i)p(x_j).$$

because the sum is over a joint probability distribution.

Example: Independence in Product of Bernoullis Model

- In a product of Bernoullis probabilities model we have

$$p(x_1, x_2, \dots, x_d) = p(x_1)p(x_2)\cdots p(x_d),$$

which we showed implies

$$p(x_i, x_j) = p(x_i)p(x_j),$$

so we have $x_i \perp x_j$ for all i and j.

- In mixture of Bernoullis x_i is not independent of x_j ($x_i \not\perp x_j$):
  - Knowing x_j tells you something about x_i.
  - But similar notation-heavy steps give the conditional independence that

$$p(x_i, x_j \mid z) = p(x_i \mid z)p(x_j \mid z),$$

"variables x_i and x_j are conditionally independent given the cluster z".

Conditional Independence

- We say that A is conditionally independent of B given C if
$$p(a, b \mid c) = p(a \mid c)p(b \mid c),$$
for all a, b, and $c \neq 0$.
- Equivalently, we have
$$p(a \mid b, c) = p(a \mid c).$$
- "If you know C, then *also* knowing B would tell you nothing about A"'.
  - In mixture of Bernoullis, given cluster there is no dependence between variables.
- We often write this as
$$A \perp B \mid C.$$
- Most models have some sort of conditional independence.
  - They were used to simplify calculations in the EM notes.
  - They determine whether message passing is efficient.

D-Separation: From Graphs to Conditional Independence

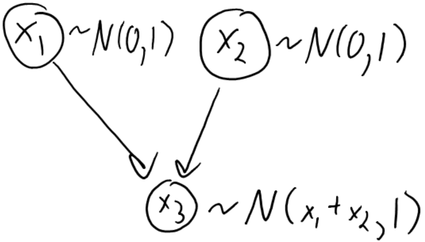
- All conditional independences implied by a DAG can be read from the graph.
- In particular: variables A and B are conditionally independent given C if:
  - "D-separation blocks all undirected paths in the graph from any variable in A to any variable in B.
- In the special of product of independent models our graph is:

- Here there are no paths to block, which implies the variables are independent.
- Checking paths in a graph tends to be faster than tedious calculations.
  - We can start connecting properties of graphs to comptuational complexity.

D-Separation as Genetic Inheritance

- The rules of d-separation are intuitive in a simple model of gene inheritance:
  - Each person has single number, which we'll call a "gene".
  - If you have no parents, your gene is a random number.
  - If you have parents, your gene is a sum of your parents plus noise.

- For example, think of something like this:



- Graph corresponds to the factorization $p(x_1, x_2, x_3) = p(x_1)p(x_2)p(x_3 \mid x_1, x_2)$.
  - Are x_1 and x_2 independent here?

D-Separation as Genetic Inheritance

- Genes of people are independent if knowing one says nothing about the other:
  - Knowing your parent's "gene" gives you information about your gene.
  - Knowing your friend's gene tells doesn't say anything about your gene.
- Genes of people can be conditionally independent given a third person:
  - Knowing your grandparent's gene tells you something about your gene.
  - But grandparent's gene isn't useful if you know parent's gene.

D-Separation Case 0 (No Paths and Direct Links)

Are genes in person x independent of the genes in person y?

- No path: x and y are not related (independent),

  We have $x \perp y$: there are no paths to be blocked.

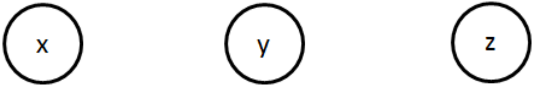
- Direct link: x is the parent of y,

  We have $x \not\perp y$: knowing x tells you about y (direct paths aren't blockable).

D-Separation Case 0 (No Paths and Direct Links)

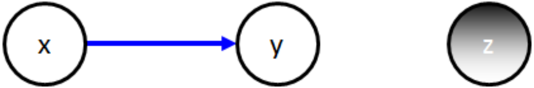
Neither case changes if we have a third independent person z:

- No path: If x and y are independent,



  We have $x \perp y$: adding z doesn't make a path.

- Direct link: x is the parent of y,

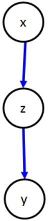


  We have $x \not\perp y \mid z$: adding z doesn't block path.

  - We use **black or shaded** nodes to denote values we condition on (in this case z).

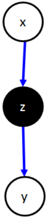
D-Separation Case 1: Chain

- Case 1: x is the grandparent of y.
  - If z is the mother we have:



We have $x \not\perp y$: knowing x would give information about y because of z

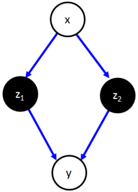
  - But if z is *observed*:



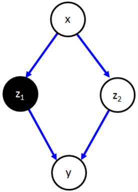
In this case $x \perp y \mid z$: knowing z "breaks" dependence between x and y.

D-Separation Case 1: Chain

- Consider weird case where parents z_1 and z_2 share parent x:
  - If z_1 and z_2 are observed we have:



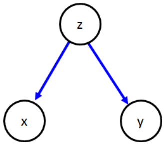
    We have $x \perp y \mid z_1, z_2$: knowing both parents breaks dependency.
  - But if only z_1 is *observed*:



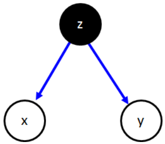
    We have $x \not\perp y \mid z_1$: dependence still "flows" through z_2.

D-Separation Case 2: Common Parent

- Case 2: x and y are sibilings.
  - If z is a common unobserved parent:



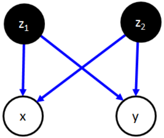
    We have $x \not\perp y$: knowing x would give information about y.
  - But if z is *observed*:



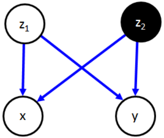
    In this case $x \perp y \mid z$: knowing z "breaks" dependence between x and y.

D-Separation Case 2: Common Parent

- Case 2: x and y are siblings.
  - If z_1 and z_2 are common observed parents:



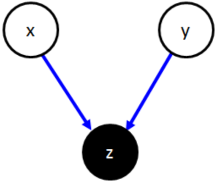
We have $x \perp y \mid z_1, z_2$: knowing z_1 and z_2 breaks dependence between x and y.
  - But if we only observe z_2:



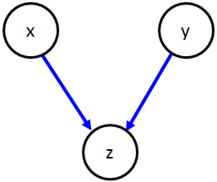
Then we have $x \not\perp y \mid z_2$: dependence still "flows" through z_1.

D-Separation Case 3: Common Child

- Case 3: x and y share a child z:
  - If we observe z then we have:



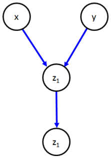
    We have $x \not\perp y \mid z$: if we know z, then knowing x gives us information about y.
  - But if z is not observed:



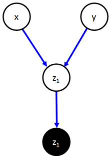
    We have $x \perp y$: if you don't observe z then x and y are independent.
- Different from Case 1 and Case 2: not observing the child blocks path.

D-Separation Case 3: Common Child

- Case 3: x and y share a child z_1:
  - If there exists an unobserved grandchild z_2:



    We have $x \perp y$: the path is still blocked by not knowing z_1 or z_2.
  - But if z_2 is observed:



    We have $x \not\perp y \mid z_2$: grandchild creates dependence even with unobserved parent.
- Case 3 needs to consider descendants of child.

Summary

- Message-passing allow efficient calculations with Markov chains.
- DAG models factorize joint distribution into product of conditionals.
  - Assume conditionals depend on small number "parents".
  - Joint distribution of models we've discussed can be written as DAG models.
- Conditional independence of A and B given C:
  - Knowing B tells us nothing about A if we already know C.
- D-separation allows us to test conditional independences based on graph.
- Next time: the IID assumption as a graphical model?

Computing Conditional Conditional Probabilities

- Previously: Monte Carlo for approximating conditional probabilities
- For Gaussian/discrete Markov chains, we can do better than rejection sampling.
  1. We can generate exact samples from conditional distribution (bonus slide).
     - Rejection sampling is not needed, relies on "backwards sampling" in time.
  2. We can find conditional decoding $\max_{x \mid x_{j'}=c} p(x)$:
     - Run Viterbi decoding with $M_{j'}(c) = 1$ and $M_{j'}(c') = 0$ for $c \neq c'$.
  3. We can find univariate conditionals, $p(x_j \mid x_{j'})$.

- Example of computing $p(x_1 = c \mid x_3 = 1)$ in a length-4 discrete Markov chain:

$$\begin{aligned} p(x_1 = c \mid x_3 = 1) &\propto p(x_1 = c, x_3 = 1) \\ &= \sum_{x_4} \sum_{x_2} p(x_1 = c, x_2, x_3 = 1, x_4), \end{aligned}$$

  where the normalizing constant is the marginal $p(x_3 = 1)$.
- This is a sum over k^{d-2} possible assignments to other variables.

Distributing Sum across Product

- Fortunately, the Markov property makes the sums simplify as before:

$$
\begin{aligned}
\sum_{x_4}\sum_{x_2} p(x_1 = c, x_2, x_3 = 1, x_4) &= \sum_{x4}\sum_{x3=1}\sum_{x_2}\sum_{x_1=c} p(x_4 \mid x_3)p(x_3 \mid x_2)p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x4}\sum_{x3=1}\sum_{x_2} p(x_4 \mid x_3)p(x_3 \mid x_2)\sum_{x_1=c} p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x4}\sum_{x3=1} p(x_4 \mid x_3)\sum_{x_2} p(x_3 \mid x_2)\sum_{x_1=c} p(x_2 \mid x_1)M_1(x_1) \\
&= \sum_{x4}\sum_{x3=1} p(x_4 \mid x_3)\sum_{x_2} p(x_3 \mid x_2)M_2(x_2) \\
&= \sum_{x4}\sum_{x3=1} p(x_4 \mid x_3)M_3(x_3) \\
&= \sum_{x4} M_4(x_4),
\end{aligned}
$$

where $M_j(x_j)$ now sums over paths ending in x_j instead of maximizing.
  - And we set $M_1(c') = 0$ if $c' \neq c$ and $M_3(c') = 0$ for $c' \neq 1$.

Conditionals via Backwards Messages

- Performing our conditional calculation using backwards messages.

$$
\begin{aligned}
\sum_{x_4}\sum_{x_2} p(x_1 = c, x_2, x_3 = 1, x_4) &= \sum_{x_1=c}\sum_{x_2}\sum_{x3=1}\sum_{x4} p(x_4 \mid x_3)p(x_3 \mid x_2)p(x_2 \mid x_1)p(x_1) \\
&= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1) \sum_{x3=1} p(x_3 \mid x_2)\sum_{x4} p(x_4 \mid x_3) \\
&= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1) \sum_{x3=1} p(x_3 \mid x_2)\sum_{x4} p(x_4 \mid x_3) \underbrace{V_4(x_4)}_{=1} \\
&= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1) \sum_{x3=1} p(x_3 \mid x_2)V_3(x_3) \\
&= \sum_{x_1=c} p(x_1)\sum_{x_2} p(x_2 \mid x_1)V_2(x_2) \\
&= \sum_{x_1=c} p(x_1)V_1(x_1).
\end{aligned}
$$

Forward-Backward Algorithm

- Generic forward and backward messages for discrete marginals have the form

$$M_j(x_j) = \sum_{x_{j-1}} p(x_j \mid x_{j-1}) M_{j-1}(x_{j-1}), \quad V_j(x_j) = \sum_{x_{j+1}} p(x_{j+1} \mid x_j) V_{j+1}(x_{j+1}).$$

- We can compute $p(x_j = c \mid x_{j'} = c')$ using only forward messages:
  - Set $M_j(c) = 1$ and $M_{j'}(c') = 1$.
- Why we would need backward messages?

Forward-Backward Algorithm

- We can compute $p(x_j = c \mid x_{j'} = c')$ for all j in $O(dk^2)$ with both messages.
- First compute all message normally with $M_{j'}(c') = 1$ and $V_{j'}(c') = 1$.

  (Other $M_{j'}$ and $V_{j'}$ are set to 0)
- We then have that
  - $M_j(x_j)$ sums up all the paths that end in state x_j (with $x_{j'} = c'$).
  - $V_j(x_j)$ sums up all the paths that start in state x_j (with $x_{j'} = c'$).
  - We can combine these values to get

$$p(x_j \mid x_{j'}) \propto M_j(x_j)V_j(x_j),$$

  - Computing all M_j and V_j is called the forward-backward algorithm.

Conditional Samples from Gaussian/Discrete Markov Chain

Generating exact conditional samples from Gaussian/discrete Markov chains:

1. If we're only conditioning on first j states, $x_{1:j}$, just fix these values and start ancestral sampling from time $(j+1)$.
2. If we have the marginals $p(x_j)$, we can get the "backwards" transition probabilities using Bayes rule,

$$p(x_j \mid x_{j+1}) = \frac{p(x_{j+1} \mid x_j)p(x_j)}{p(x_{j+1})},$$

   which lets us run ancestral sampling in reverse: sample x_d from $p(x_d)$, then x_{d-1} from $p(x_{d-1} \mid x_d)$, and so on.
3. If we're only conditioning on last j states $x_{d-j:d}$, run CK equations to get marginals and then start ancestral sampling "backwards" starting from $(d-j-1)$ to sample the earlier states.

Conditional Samples from Gaussian/Discrete Markov Chain

4. If we're conditioning on contiguous states in the middle, $x_{j:j'}$, run ancestral sampling forward starting from position $(j'+1)$ and backwards starting from position $(j-1)$.
5. If you condition on non-contiguous positions j and j' with $j < j'$, need to do (i) forward sampling starting from $(j'+1)$, (ii) backward sampling starting from $(j-1)$, and (iii) CK equations on the sequence $(j:j')$ to get marginals conditioned on value of j then backwards sampling back to j starting from $(j'-1)$.

The above are all special cases of conditioning in an undirected graphical model (UGM), followed by applying the "forward-filter backward-sampling" algorithm on each of the resulting chain-structured UGMs.