cs542g Final Exam
due December 22, 2009

This is a take-home exam. When you feel ready to take it, set aside **three hours** to write it, writing your solutions to the six questions on paper. After the three hours are up, stop writing, and relay your solutions to me (either scanned and emailed, or hardcopy in my mailbox or under my door or given to me in person).

This is open book in the following sense: before you start you can gather your notes, print-outs, books you may like, or other materials. However, while you are writing (in particular, as soon as you have seen the questions) you are not allowed to search out additional reference material—in particular, no help from the internet or other people.

If a question is unclear, do your best to state what puzzles you and what you will assume to proceed with writing your answer.

Attempt all questions. Partial marks will be awarded for demonstrating understanding of the relevant material even if you can't fully solve the problem.

1 . . . .

The Richardson iteration for solving an SPD system of equations $Ax = b$ starts from any initial guess x_0 and constructs new guesses using a predetermined scalar parameter ω:

- Compute the current residual $r_i = b - Ax_i$
- Update to get the next guess as $x_{i+1} = x_i + \omega r_i$

Of critical importance is choosing a good parameter ω. Work out a condition on ω that guarantees convergence to the solution of the linear system. (Hint: look at how the residual r_{i+1} relates to the previous residual r_i.) How fast is the convergence in terms of properties of the matrix A?

2 . . . .

Propose an algorithm for efficiently approximating the condition number of a large, sparse SPD matrix A, assuming that computing and storing the inverse (which is typically fully dense even if A is sparse) is far too expensive to be possible.

3 . . . .

Show why Rayleigh Quotient Iteration is expected to have cubic convergence. In particular, assume the current guess x at eigenvector u_1 for eigenvalue λ_1 is of the form

$$x = u_1 + \epsilon u_2$$

where u_2 is an eigenvector for eigenvalue $\lambda_2 \neq \lambda_1$ and $|\epsilon| < 1$. Show the speed of convergence is independent of the eigenvalues λ_1 and λ_2.

4 . . . .

Derive an algorithm for computing the square root of a positive number y with just the regular arithmetic operations, using Newton applied to

$$\min_x \frac{1}{3}x^3 - xy$$

Prove that it always converges if you start with $x = y$ as the initial guess.

5 . . . .

Consider the n-body problem, in particular evaluating the potential of a cluster of n points at the origin. For simplicity, assume the points all lie on the positive x-axis so we can ignore the other coordinates, and that all masses are equal, so we are left with evaluating

$$\sum_{i=1}^{n} \frac{1}{x_i}$$

Further assume their centre of mass is at location D and the cluster has radius r, i.e. $D - r \leq x_i \leq D + r$ for all i. The basic approximation is to use n/D.

Now consider two ways to improve the accuracy with a little more work. One would be to split the cluster into two halves, which we will assume both have $n/2$ points and radius $r/2$. Another would be to include one more term in the Taylor series expansions underlying the approximation. Compare the gain in accuracy and extra cost of each approach.

6 ....

Take a simple 1D Poisson problem:

$$\begin{aligned} \frac{d^2}{dx^2}u &= f \qquad \text{for } 0 < x < 1 \\ u(0) &= a \\ u(1) &= b \end{aligned}$$

We can set up a discretization with grid points $x_i = i\Delta x$ for some spacing Δx. Derive a one-sided finite difference approximation to d^2u/dx^2 at x_i from the Taylor series expansion of $u(x_{i+1})$ and $u(x_{i+2})$ around the point $u(x_i)$.

Will this finite difference method work for the problem? Hint: figure out the structure of the matrix—it's definitely not symmetric.