Notes

- Preemptive make-up lecture this week
- Assignment 2 due by tomorrow morning
- Assignment 3 coming soon

Aside: spectral methods

- Last time we constructed finite difference methods for solving
$$\nabla\bullet\nabla u = f$$
on a rectangular domain
- Can do better, using the Fourier Transform
$$u(x) = \sum_{i=-\infty}^{\infty} \hat{u}_i e^{\sqrt{-1}(2\pi i x)}$$
- Gives the "**spectral method**"
  - O(N log N) with FFT
  - Converges as fast as the solution is smooth

Getting off regular grids

- Problems with finite differences (and spectral methods)
  - Matching non-rectangular geometry
  - Adaptivity
- Not much satisfactory progress in using Taylor series approach off regular grids\
- Alternative (as at the start of the course): pick a set of basis functions, assume solution is in the span
$$u(\vec{x}) = \sum_{j=1}^{n} u_j \phi_j(\vec{x})$$

Collocation

- The original PDE: $\nabla\bullet\nabla u = f$
  - (worry about boundary conditions later)
- Plugging in our form of the solution won't be exact everywhere (in general)
- Can just force it to be true at n points: "**collocation**" method
$$\sum_{j=1}^{n} u_j \left(\nabla\bullet\nabla\phi_j(\vec{x}_i)\right) = f(\vec{x}_i) \quad i = 1,\ldots,n$$

Problems

- Collocation works fine for smooth basis functions, such as RBF's
  - But RBF's have issues at boundaries, dealing with non-smooth conditions, global support / dense matrices
  - Can still be made to work very well!
- Constructing smooth-enough and compactly-supported functions on a mesh isn't so attractive
  - Need to use e.g. subdivision schemes, still issues at some nodes

Galerkin FEM

- FEM = Finite Element Method
- Includes collocation approach, but more commonly associated with **Galerkin** approach
- Rephrase PDE from "strong" form (equation holds true at all x)
to "weak" form:
$$\int_\Omega \left(\nabla\bullet\nabla u(\vec{x}) - f(\vec{x})\right)\psi(\vec{x}) = 0 \quad \forall\psi \in W$$
- Functions ψ are called "test functions"

Galerkin continued

- Integrate by parts:

$$\int_{\Omega} (\nabla \bullet \nabla u - f)\psi = \int_{\Omega} (-\nabla u \bullet \nabla \psi - f\psi) + \oint_{\partial\Omega} \psi \nabla u \bullet \hat{n}$$

  - Ignore boundary term for now…
- Reduced from two derivatives to one!
- Can now choose u (and test functions) to be less smooth
  - In fact, since we're integrating, not evaluating at a point, u and test functions don't need derivative absolutely everywhere…

Galerkin (finally)

- Galerkin method: pick u and test functions from the same finite dimensional space

$$u(\vec{x}) = \sum_{j=1}^{n} u_j \phi_j(\vec{x}), \quad \psi_i = \phi_i$$

- Get n equations for the n unknowns: (ignoring boundary for the moment)

$$-\int_{\Omega} \nabla \left(\sum_{j=1}^{n} u_j \phi_j(\vec{x}) \right) \bullet \nabla \phi_i(\vec{x}) - f(\vec{x}) \phi_i(\vec{x}) \, dx = 0$$

The Equations

- Rearranging, we get:

$$-\sum_{j=1}^{n} \left(\int_{\Omega} \nabla \phi_i \bullet \nabla \phi_j \right) u_j = \int_{\Omega} f \phi_i$$

$$-Au = f$$

- Built-in properties:
  - A is symmetric
  - A is positive semi-definite
  - (neither is true necessarily for collocation)

Boundary Conditions

- Recall boundary term in weak form:

$$\int_{\Omega} (\nabla \bullet \nabla u - f)\psi = -\int_{\Omega} (\nabla u \bullet \nabla \psi - f\psi) + \oint_{\partial\Omega} \psi \nabla u \bullet \hat{n}$$

- For Dirichlet boundary conditions need to restrict u=g at boundary and test function =0 $\quad u = g$
- For Neumann boundary conditions allow test function to be nonzero and get extra term in equations $\quad \nabla u \bullet \hat{n} = h$

Picking a FEM space

- Simplest space with "weak" first derivatives: continuous, piecewise linear
- Define values at mesh vertices
- Linearly interpolate across each interval / triangle / tetrahedron
- **Nodal basis functions**: 1 at a mesh vertex, zero at the others "hat functions"