Notes

- Added required reading to web (numerical disasters)

Interpolation in 1D

- Linear interpolation
- Polynomial interpolation
  - Lagrange formula
  - Dangers
- Splines and more
- Can be extended to 2d etc with tensor products

Function Space Interpretation

- Rather than think of it as a discrete problem:

  "How do I estimate function at a given point given data at nearby points"

  think in terms of function space:

  "Which function from my chosen space fits the data?"

- This perspective will come up again and again

1D Function Spaces

- Piecewise-linear interpolation
- Polynomial interpolation
- Splines
- Choice of basis

Scattered Data Interpolation

- Look at something more interesting
- Arbitrarily scattered data points in multiple dimensions
- How do we fit a smooth surface through them?

- Triangulation
- Radial Basis Functions
- Moving Least-Squares

Triangulation

- First construct a mesh through the data points
- Then interpolate on each triangle separately
- In 2D this works fine
- In 3D (tetrahedralization) it gets tricky
- Doesn't scale well past 3D

- We'll talk about mesh generation later

Radial Basis Functions (RBF)

- Assume data is in the span of a set of radial basis functions
- Solve for the right coefficients

RBF + polynomials

- Problem: RBF's can't even get constants correct
- Fix by adding a low order polynomial term
- Under-constrained system!
- What extra constraints should we add?

PDE Interpretation

- Think of the problem physically: what problem are we really solving? (another perspective which will come up again and again)
- We want the smoothest surface which goes through the given points
- Define smoothest, then solve the problem exactly (gives a differential equation…)

First Try in 1D

- Try to minimize a measure of how 'wobbly' the function is:

$$\min_{f(x_i)=f_i} \int_{-\infty}^{\infty} \tfrac{1}{2}\left(f'(x)\right)^2 dx$$

- Calculus of variations gives:

$$f''(x) = 0 \quad \text{in } (x_{i-1}, x_i)$$
$$f(x_i) = f_i$$

Exact Solution

- Direct method: differential equation has piecewise linear solution
  - But this is too hard to do directly in more dimensions…
- Method of fundamental solution:
  - First find a function that satisfies

$$\phi''(x) = \delta(x)$$

  - Simplest (symmetric) answer is:

$$\phi(x) = \tfrac{1}{2}|x|$$

Fundamental Solution…

- Then we know the exact solution is of the form

$$f(x) = A + \sum_{i=1}^{n} \lambda_i \phi(x - x_i)$$
$$= A + \sum_{i=1}^{n} \tfrac{1}{2}\lambda_i |x - x_i|$$

- The constant A doesn't affect f'
- Also want f' → 0 at infinity
- Away from all data points, f' is proportional to $\sum_{i=1}^{n} \lambda_i$

Simple 1D RBF

- Putting this together, our equations are then:

$$A + \sum_{i=1}^{n} \lambda_i \phi(x_j - x_i) = f_i \quad j = 1, \ldots, n$$

$$\sum_{i=1}^{n} \lambda_i = 0$$

- n+1 linear equations for n+1 unknowns

Smoother Interpolation

- Minimize curvature

$$\min_{f(x_i)=f_i} \int_{-\infty}^{\infty} \tfrac{1}{2}\left(f''(x)\right)^2 dx$$

- Calculus of variations gives us:

$$f^{(iv)}(x) = 0 \quad \text{in } (x_{i-1}, x_i)$$
$$f(x_i) = f_i$$

Smoother Fundamental Solution

- Our basis function is now

$$\phi(x) = \tfrac{1}{12}|x|^3$$

  and we also include a linear term in polynomial (doesn't change f'')
- So our solution is of the form:

$$f(x) = A + Bx + \sum_{i=1}^{n} \lambda_i \phi(x - x_i)$$

Boundedness

- Outside of the data points, f'' is:

$$\begin{aligned} f''(x) &= \pm \sum \tfrac{1}{2} \lambda_i (x - x_i) \\ &= \pm \tfrac{1}{2}\left(\sum \lambda_i\right) x \mp \tfrac{1}{2}\left(\sum \lambda_i x_i\right) \end{aligned}$$

- Setting this to zero gives two conditions on the coefficients…
- n+2 equations for n+2 unknowns

More Dimensions

- Same approach generalizes
- E.g. "thin-plate spline" comes out of

$$\min_{f(x_i)=f_i} \int \tfrac{1}{2}\left(\nabla \bullet \nabla f\right)^2$$

- The Laplacian $\nabla \bullet \nabla$ is a measure of mean curvature
- Calculus of variations gives the "biharmonic equation":

$$\nabla \bullet \nabla \; \nabla \bullet \nabla f = 0$$

Thin-Plate Spline

- In 2D, fundamental solution is proportional to

$$\phi(x) = |x|^2 \log|x|$$

  and in 3D: $\phi(x) = |x|$
- Include a linear polynomial
- Boundedness conditions are:

$$\sum \lambda_i = 0, \quad \sum \lambda_i x_i = 0, \quad \sum \lambda_i y_i = 0, \quad \sum \lambda_i z_i = 0$$

Other RBF's

- Other basis functions can be used of course
- Usual alternatives:
  - Triharmonic basis function: $\phi(x) = |x|^3$
  - Multiquadric: $\phi(x) = \sqrt{|x|^2 + c^2}$
  - Gaussian: $\phi(x) = \exp\left(-\frac{x^2}{c^2}\right)$
- With or without low order polynomial

The Equations

- In all cases, we end up with a linear system to solve
- How do we solve it?
- Gaussian Elimination is the usual answer