Notes

- In assignment 1, problem 2: smoothness = number of times differentiable

The Symmetric Eigenproblem

- Assume A is symmetric and real
- Find orthogonal matrix V and diagonal matrix D s.t. AV=VD
  - Diagonal entries of D are the eigenvalues, corresponding columns of V are the eigenvectors
- Also put: $A=VDV^T$ or $V^TAV=D$
- There are a few strategies
  - More if you only care about a few eigenpairs, not the complete set…
- Also: finding eigenvalues of an nxn matrix is equivalent to solving a degree n polynomial
  - No "analytic" solution in general for n≥5
  - Thus general algorithms are **iterative**

The Jacobi Algorithm

- One of the oldest numerical methods, but still of interest
- Start with initial guess at V (e.g. V=I), set $A=V^TAV$
- For k=1, 2, …
  - If A is close enough to diagonal (Frobenius norm of off-diagonal tiny relative to A) stop
  - Find a Givens rotation Q that solves a 2x2 subproblem
    - Either zeroing max off-diagonal entry, or sweeping in order through matrix.
  - $V=VQ$, $A=Q^TAQ$
- Typically O(log n) sweeps, $O(n^2)$ rotations each
- Quadratic convergence in the limit

The Power Method

- Start with some random vector v, $\|v\|_2=1$
- Iterate $v=(Av)/\|Av\|$
- What happens? How fast?

Shift and Invert (Rayleigh Iteration)

- Say we know the eigenvalue we want is approximately λ_k
- The matrix $(A-\lambda_k I)^{-1}$ has the same eigenvectors as A
- But the eigenvalues are $\mu = \dfrac{1}{\lambda - \lambda_k}$
- Use this in the power method instead
- Even better, update guess at eigenvalue each iteration: $\lambda_{k+1} = v_{k+1}^T A v_{k+1}$
- Gives cubic convergence! (triples the number of significant digits each iteration when converging)

Maximality and Orthogonality

- Unit eigenvectors v_1 of the maximum magnitude eigenvalue satisfy

$$\|Av_1\|_2 = \max_{\|u\|=1} \|Au\|_2$$

- Unit eigenvectors v_k of the k'th eigenvalue satisfy

$$\|Av_k\|_2 = \max_{\substack{\|u\|=1 \\ u^T v_i = 0, i<k}} \|Au\|_2$$

- Can pick them off one by one, or….

Orthogonal iteration

- Solve for lots (or all) of eigenvectors simultaneously
- Start with initial guess V
- For k=1, 2, …
  - Z=AV
  - VR=Z (QR decomposition: orthogonalize W)
- Easy, but slow
  (linear convergence, nearby eigenvalues slow things down a lot)

Rayleigh-Ritz

- Before going further with full problem, take a look at intermediate problem:
  find a subset of the eigenpairs
  - E.g. largest k, smallest k
- Orthogonal estimate V (nxk) of eigenvectors
- Simple Rayleigh estimate of eigenvalues:
  - diag(V^TAV)
- Rayleigh-Ritz approach:
  - Solve kxk eigenproblem V^TAV
  - Use those eigenvalues (Ritz values) and the associated orthogonal combinations of columns of V
  - Note: another instance of "assume solution lies in span of a few basis vectors, solve reduced dimension problem"

Solving the Full Problem

- Orthogonal iteration works, but it's slow
- First speed-up: make A tridiagonal
  - Sequence of symmetric Householder reflections
  - Then Z=AV runs in $O(n^2)$ instead of $O(n^3)$
- Other ingredients:
  - Shifting: if we shift A by an exact eigenvalue, $A-\lambda I$, we get an exact eigenvector out of QR (improves on linear convergence)
  - Division: once an offdiagonal is almost zero, problem separates into decoupled blocks