Notes

- Required reading:
  - Baraff & Witkin, "Large steps in cloth animation", SIGGRAPH'98
  - Grinspun et al., "Discrete shells", SCA'03
  - Bridson et al., "Simulation of clothing with folds and wrinkles", SCA'03

1D Elastic Continuum

- From last class: elastic rod
  - linear density "rho" (not necessarily constant)
  - Young's modulus E (not necessarily constant)
  - Paratemerized by p

$$\ddot{x}(p) = \frac{1}{\rho}\frac{\partial}{\partial p}\left(E(p)\left(\frac{\partial}{\partial p}x(p) - 1\right)\right)$$

- If homogenous, simplifies to:

$$\frac{\partial^2 x}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 x}{\partial p^2}$$

Sound waves

- Try solution $x(p,t)=x_0(p-ct)$
- And $x(p,t)=x_0(p+ct)$
- So speed of "sound" in rod is $\sqrt{\frac{E}{\rho}}$
- Courant-Friedrichs-Levy (CFL) condition:
  - Numerical methods only will work if information transmitted numerically at least as fast as in reality (here: the speed of sound)
  - Usually the same as stability limit for good explicit methods [what are the eigenvalues here]
  - Implicit methods transmit information infinitely fast

Why?

- Are sound waves important?
  - Visually? Usually not
- However, since speed of sound is a material property, it can help us get to higher dimensions
- Speed of sound in terms of one spring (using linear density m/L) is

$$c = \sqrt{\frac{kL}{m}}$$

- So in higher dimensions, just pick k so that c is constant
  - m is mass around spring [triangles, tets]
  - Optional reading: van Gelder

Potential energy

- Another way to look at the elastic spring forces: through potential energy
- Recall for a system at position(s) x, potential energy W(x) gives the force

$$F = -\frac{\partial W}{\partial x}$$

- For example, this gives conservation of total energy K+E: $\frac{d}{dt}\left(\tfrac{1}{2}v^T M v + W\right) = v^T M \frac{dv}{dt} + \frac{\partial W}{\partial x}\frac{dx}{dt}$

$$\begin{aligned} &= v^T M a - F^T v \\ &= v^T F - v^T F \\ &= 0 \end{aligned}$$

Spring potential

- For a single spring,

$$W_{ij} = \tfrac{1}{2} k_{ij} \left(\frac{|x_i - x_j|}{L_{ij}} - 1 \right)^2 L_{ij}$$

  - Note we're squaring the percent deformation (so this always increases as we move away from undeformed), and scaling by the strength of spring and by the length (amount of material it represents)
- To get the force on i, differentiate w.r.t. x_i:

$$f_{ij} = -\frac{\partial W_{ij}}{\partial x_i} = -k_{ij} \left(\frac{|x_i - x_j|}{L_{ij}} - 1 \right) \frac{x_i - x_j}{|x_i - x_j|}$$

1D Continuum potential

- Add up the potential energies for each spring to approximate the total potential energy for the elastic rod:

$$W \approx \sum_i \tfrac{1}{2} E_{i+½} \left(\frac{x_{i+1} - x_i}{p_{i+1} - p_i} - 1 \right)^2 (p_{i+1} - p_i)$$

- Take the limit as Δp goes to zero:

$$W = \int_0^1 \tfrac{1}{2} E(p) \left(\frac{\partial x}{\partial p} - 1 \right)^2 dp$$

- Now: how do we get forces out of this? The negative gradient of W w.r.t. x(p)?

Directional derivatives (regular calculus)

- Pick a direction, or test vector, q
- Direction derivative along q is:

$$D_q W(x) = \lim_{\varepsilon \to 0} \frac{W(x + \varepsilon q) - W(x)}{\varepsilon}$$

- Or alternatively

$$D_q W(x) = g'(0) \quad where \quad g(\varepsilon) = W(x + \varepsilon q)$$

- And the gradient ∂W/∂x is the vector s.t.

$$D_q W(x) = \frac{\partial W}{\partial x} \bullet q \quad \forall q$$

The variational derivative

- We want to take the "gradient" of a continuum potential energy to get the force density:

$$f = -\frac{\partial W[x]}{\partial x} \quad where \quad W[x] = \int_0^1 \tfrac{1}{2} E\left(\frac{\partial x}{\partial p} - 1\right)^2 dp$$

- But how do you differentiate w.r.t. a function x(p)?
- Let's first look at a directional derivative: look at the energy at x+εq
  - q is a direction, or test function

$$g(\varepsilon) = W[x + \varepsilon q] = \int_0^1 \tfrac{1}{2} E\left(\frac{\partial x}{\partial p} + \varepsilon\frac{\partial q}{\partial p} - 1\right)^2 dp$$

The variational derivative 2

- Now differentiate w.r.t. the scalar:

$$g'(\varepsilon) = \frac{d}{d\varepsilon}\int_0^1 \tfrac{1}{2} E\left(\frac{\partial x}{\partial p} + \varepsilon\frac{\partial q}{\partial p} - 1\right)^2 dp$$

$$= \int_0^1 \tfrac{1}{2} E\frac{d}{d\varepsilon}\left(\left(\frac{\partial x}{\partial p} - 1\right)^2 + 2\varepsilon\left(\frac{\partial x}{\partial p} - 1\right)\frac{\partial q}{\partial p} + \varepsilon^2\frac{\partial q}{\partial p}^2\right) dp$$

$$= \int_0^1 E\left(\left(\frac{\partial x}{\partial p} - 1\right)\frac{\partial q}{\partial p} + \varepsilon\frac{\partial q}{\partial p}^2\right) dp$$

- And evaluate at 0 to get the directional derivative

$$D_q W[x] = g'(0) = \int_0^1 E\left(\frac{\partial x}{\partial p} - 1\right)\frac{\partial q}{\partial p} dp$$

The variational derivative 3

- We want to make it look like an inner-product of the "gradient" with q()
  - Use integration by parts:

$$D_q W[x] = -\int_0^1 \frac{\partial}{\partial p}\left(E\left(\frac{\partial x}{\partial p} - 1\right)\right) q\, dp + \left[E\left(\frac{\partial x}{\partial p} - 1\right) q\right]_0^1$$

- Ignoring boundary conditions for now, we see that the variational derivative of W at some interior point p is just:

$$-\frac{\partial}{\partial p}\left(E\left(\frac{\partial x}{\partial p} - 1\right)\right)$$

Force density

- The elastic force density is the negative of the variational derivative at a point:

$$f = \frac{\partial}{\partial p}\left(E\left(\frac{\partial x}{\partial p} - 1\right)\right)$$

- The acceleration of that point is force density divided by mass density:

$$\frac{\partial^2 x}{\partial t^2} = \frac{1}{\rho}\frac{\partial}{\partial p}\left(E\left(\frac{\partial x}{\partial p} - 1\right)\right)$$

- Which is exactly what we got before!

Discretized potential

- Now we have an alternative way to discretize the equation:
  - Approximate the potential energy integral with a discrete sum
  - Take the gradient to get forces
- This approach generalizes to all sorts of forces
- Let's do it for multi-dimensional springs

Multi-dimensional spring potential

- Change the L scale in the 1D spring potential to be the area/volume around the spring
  - When we add up, we get an approximation of an integral over the elastic object

$$W_{ij} = \tfrac{1}{2} E \left(\frac{|x_i - x_j|}{L_{ij}} - 1 \right)^2 A_{ij}$$

- Then get the spring force on i due to j:

$$f_{ij} = -\frac{\partial W_{ij}}{\partial x_i} = -E \frac{A_{ij}}{L_{ij}} \left(\frac{|x_i - x_j|}{L_{ij}} - 1 \right) \frac{x_i - x_j}{|x_i - x_j|}$$

Bending

- For simulating cloth and thin shells, also need forces to resist bending
- Nontrivial to directly figure out such a force on a triangle mesh
- Even harder to make sure it's roughly mesh-independent
- So let's attack the problem with a potential energy formulation

Bending energy

- Integrate mean curvature squared over the surface:

$$W_{bend} = \iint_{\Omega} \tfrac{1}{2} B \kappa^2$$

- Let's bypass the continuum formulation, and jump to a discrete approximation of this integral
- Split mesh up into regions around "hinges" (common edges between triangles)

$$W_{bend} \approx \sum_{e:\text{edge}} \tfrac{1}{2} B_e \kappa_e^2 A_e$$

  - At each interior edge e, have bending stiffness B_e, curvature estimate κ_e and area of region A_e

Edge curvature estimate

- Look at how the hinge is bent
  - Dihedral angle between the incident triangles
- Think of fitting a cylinder of radius R parallel to edge, mean curvature is 1/(2R)
  - [side-view of triangle pair: angle between normals is theta, triangle altitudes are h_1 and h_2]
- For small bend angles (limit as mesh is refined!)

$$\frac{1}{2R} \approx \frac{\theta}{h_1 + h_2}$$

Discrete bending energy

- As a rough approximation then, using $A_e = |e|(h_1+h_2)/6$:

$$W_{bend} \approx \sum_{e:\text{edge}} \tfrac{1}{2} B_e \left(\frac{\theta |e|}{6A_e} \right)^2 A_e$$

$$= \sum_{e:\text{edge}} \tfrac{1}{2} B_e \frac{|e|^2}{36 A_e} \theta^2$$

- Treat all terms except theta as constant (measured from initial mesh), differentiate that w.r.t. positions x

Bending forces

- See e.g. Bridson et al. “Simualtion of clothing…” for reasonable expressions for forces
  - Additional simplification: replace theta with a similarly-behaved trig function that can be directly computed
  - Warning: derivation is quite different!