

What is Poisson's ratio?

- ◆ Has to be between -1 and 0.5
- ◆ 0.5 is exactly incompressible
 - [derive]
- Negative is weird, but possible [origami]
- ♦ Rubber: close to 0.5
- ◆ Steel: more like 0.33
- Metals: usually 0.25-0.35
- What should cork be?

Putting it together

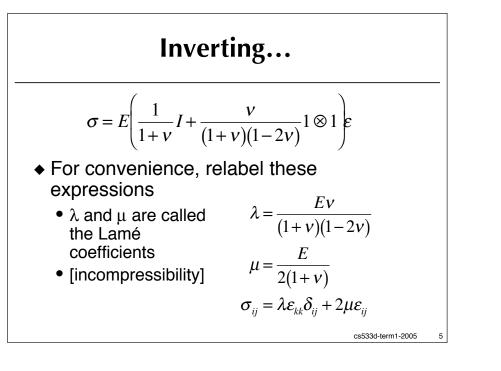
$$E\varepsilon_{11} = \sigma_{11} - v\sigma_{22} - v\sigma_{33}$$
$$E\varepsilon_{22} = -v\sigma_{11} + \sigma_{22} - v\sigma_{33}$$
$$E\varepsilon_{33} = -v\sigma_{11} - v\sigma_{22} + \sigma_{33}$$

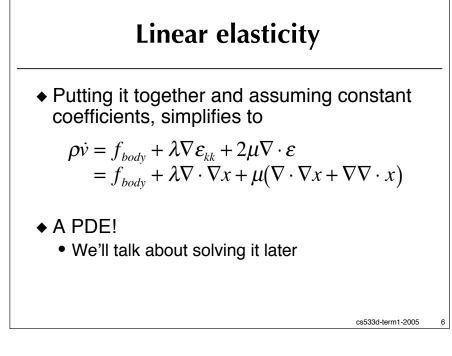
- Can invert this to get normal stress, but what about shear stress?
 - Diagonalization...
- When the dust settles,

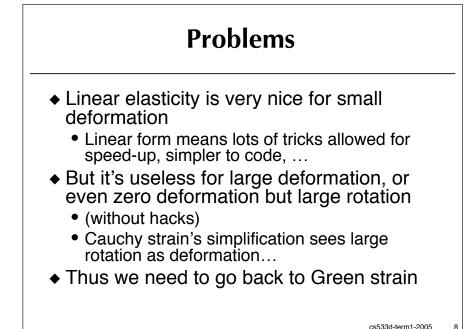
$$E\varepsilon_{ij} = (1+v)\sigma_{ij} \quad i \neq j$$

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(Almost) Linear Elasticity

- Use the same constitutive model as before, but with Green strain tensor
- This is the simplest general-purpose elasticity model
- Animation probably doesn't need anything more complicated
 - Except perhaps for dealing with incompressible materials

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2D Elasticity

- Let's simplify life before starting numerical methods
- The world isn't 2D of course, but want to track only deformation in the plane
- Have to model why
 - Plane strain: very thick material, ε₃.=0 [explain, derive σ₃.]
 - Plane stress: very thin material, σ₃.=0 [explain, derive ε₃. and new law, note change in incompressibility singularity]

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Finite Volume Method

- Simplest approach: finite volumes
 - We picked arbitrary control volumes before
 - Now pick fractions of triangles from a triangle mesh
 - Split each triangle into 3 parts, one for each corner
 - E.g. Voronoi regions
 - Be consistent with mass!
 - Assume A is constant in each triangle (piecewise linear deformation)
 - [work out]
 - Note that exact choice of control volumes not critical constant times normal integrates to zero

Finite Element Method

- #1 most popular method for elasticity problems (and many others too)
- FEM originally began with simple idea:
 - Can solve idealized problems (e.g. that strain is constant over a triangle)
 - Call one of these problems an element
 - Can stick together elements to get better approximation
- Since then has evolved into a rigourous mathematical algorithm, a general purpose black-box method
 - Well, almost black-box...

Modern Approach

- υ Galerkin framework (the most common)
- $\upsilon~$ Find vector space of functions that solution (e.g. X(p)) lives in
 - E.g. bounded weak 1st derivative: H¹
- υ Say the PDE is L[X]=0 everywhere ("strong")
- υ The "weak" statement is ∫ Y(p)L[X(p)]dp=0 for every Y in vector space
- υ Issue: L might involve second derivatives
 - E.g. one for strain, then one for div sigma
 - So L, and the strong form, difficult to define for H^1
- υ Integration by parts saves the day

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Weak Momentum Equation

- Ignore time derivatives treat acceleration as an independent quantity
 - We discretize space first, then use "method of lines": plug in any time integrator

$$L[X] = \rho \ddot{X} - f_{body} - \nabla \cdot \sigma$$

$$\int_{\Omega} YL[X] = \int_{\Omega} Y(\rho \ddot{X} - f_{body} - \nabla \cdot \sigma)$$

$$= \int_{\Omega} Y\rho \ddot{X} - \int_{\Omega} Yf_{body} - \int_{\Omega} Y\nabla \cdot \sigma$$

$$= \int_{\Omega} \Omega Y\rho \ddot{X} - \int_{\Omega} \Omega Yf_{body} + \int_{\Omega} \sigma \nabla Y$$
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Making it finite

- The Galerkin FEM just takes the weak equation, and restricts the vector space to a finitedimensional one
 - E.g. Continuous piecewise linear constant gradient over each triangle in mesh, just like we used for Finite Volume Method
- This means instead of infinitely many test functions Y to consider, we only need to check a finite basis
- The method is defined by the basis
 - Very general: plug in whatever you want polynomials, splines, wavelets, RBF's, ...

Linear Triangle Elements

- Simplest choice
- Take basis {\u03c6_i} where
 \u03c6_i(p)=1 at p_i and 0 at all the other p_i's
 - It's a "hat" function
- υ Then X(p)= $\sum_i x_i \phi_i(p)$ is the continuous piecewise linear function that interpolates particle positions
- υ Similarly interpolate velocity and acceleration
- υ Plug this choice of X and an arbitrary Y= ϕ_j (for any j) into the weak form of the equation
- υ Get a system of equations (3 eq. for each j)

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The equations

$$\int_{\Omega} \phi_j \sum_i \rho \ddot{x}_i \phi_i - \int_{\Omega} \phi_j f_{body} + \int_{\Omega} \sigma \nabla \phi_j = 0$$

$$\sum_i \int_{\Omega} \rho \phi_j \phi_i \ddot{x}_i = \int_{\Omega} \phi_j f_{body} - \int_{\Omega} \sigma \nabla \phi_j$$

•Note that ϕ_j is zero on all but the triangles surrounding it so integrals simplify

surrounding j, so integrals simplify •Also: naturally split integration into separate triangles

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