Notes

- More reading on web site
  - Baraff & Witkin's classic cloth paper
  - Grinspun et al. on bending
  - Optional: Teran et al. on FVM in graphics

Hyper-elasticity

- Another common way to look at elasticity
  - Useful for handling weird nonlinear compressibility laws, for reduced dimension models, and more
- Instead of defining stress, define an elastic potential energy
  - Strain energy density W=W(A)
  - W=0 for no deformation, W>0 for deformation
  - Total potential energy is integral of W over object
- This is called hyper-elasticity or Green elasticity
- For most (the ones that make sense) stress-strain relationships can define W
  - E.g. linear relationship: $W=\sigma:\varepsilon=\mathrm{trace}(\sigma^T\varepsilon)$

Variational Derivatives

- Force is the negative gradient of potential
  - Just like gravity
- What does this mean for a continuum?
  - $W=W(\partial X/\partial p)$, how do you do $-d/dX$?
- Variational derivative:
  - So variational derivative is $-\nabla\cdot\partial W/\partial A$
  - And $f=\nabla\cdot\partial W/\partial A$
  - Then stress is $\partial W/\partial A$

$$W_{total}[X+\varepsilon Y]=\int W\left(\frac{\partial X}{\partial p}+\varepsilon\frac{\partial Y}{\partial p}\right)$$
$$\approx\int W\left(\frac{\partial X}{\partial p}\right)+\varepsilon\frac{\partial W}{\partial A}\frac{\partial Y}{\partial p}$$
$$=W_{total}+\varepsilon\int\frac{\partial W}{\partial A}\frac{\partial Y}{\partial p}$$
$$=W_{total}-\varepsilon\int Y\nabla\cdot\frac{\partial W}{\partial A}$$

Numerics

- Simpler approach: find discrete W_{total} as a sum of W's for each element
  - Evaluate just like FEM, or any way you want
- Take gradient w.r.t. positions $\{x_i\}$
  - Ends up being a Galerkin method
- Also note that an implicit method might need Jacobian = negative Hessian of energy
  - Must be symmetric, and at least near stable configurations must be negative definite

Curve / Springs

- Take $W(A)=1/2\ E(|A|-1)^2\ L$ for each segment
  - Note factor of L: this is approximation to an integral over segment in object space of length L
- $A=(x_{i+1}-x_i)/L$ is the deformation gradient for piecewise linear elements
- Then take derivative w.r.t. x_i to get this element's contribution to force on i
- Lo and behold [exercise] get exactly the original spring force
- Note: defining stress and strain would be more complicated, because of the dimension differences
  - A is 3x1, not square

Surface elasticity

- For linear stress-strain, can use $W(A)=\sigma:G=\sigma_{ij}G_{ij}$
- The simplest model from before gives $W=\lambda G_{kk}^2+\mu G_{ij}G_{ij}$
- Remember $G=1/2(A^TA-I)$
- Tedious to differentiate, but doable
  - Tensors and chain rule over and over
- Let's leave it that
  - In practice, springs with speed-of-sound heuristic are good enough most of the time

Bending energy

- Bending is very difficult to get a handle on without variational approach
- Bending strain energy density: $W=1/2\ B\ \kappa^2$
- Here κ is mean curvature
  - Look at circles that fit surface
  - Maximum radius R and minimum radius r
  - $\kappa=(1/R + 1/r)/2$
  - Can define directly from second derivatives of X(p)
  - Uh-oh - second derivatives? [FEM nastier]
  - W is 2nd order, stress is 3rd order, force is 4th order derivatives!

Fourth order problems

- Linearize and simplify drastically, look for steady-state solution (F=0): spline equations
  - Essentially 4th derivatives are zero
  - Solutions are (bi-)cubics
- Model (nonsteady) problem: $x_{tt}=-x_{pppp}$
  - Assume solution $x(p,t) = e^{\sqrt{-1}k(p-ct)}$

    Wave of spatial frequency k, moving at speed c
  - [solve for wave parameters]
  - Dispersion relation: small waves move really fast
  - CFL limit (and stability): for fine grids, BAD
  - Thankfully, we rarely get that fine

Implicit/Explicit Methods

- Implicit bending is painful
- In graphics, usually unnecessary
  - Dominant forces on the grid resolution we use tend to be the 2nd order terms: stretching etc.
- But nice to go implicit to avoid time step restriction for stretching terms
- No problem: treat some terms (bending) explicitly, others (stretching) implicitly
  - $v_{n+1}=v_n+\Delta t/m(F_1(x_n,v_n)+F_2(x_{n+1},v_{n+1}))$
  - All bending is in F_1, half the elastic stretch in F_1, half the elastic stretch in F_2, all the damping in F_2

Discrete Mean Curvature

- [draw triangle pair]
- κ for that chunk varies as $\kappa \sim \dfrac{\theta}{h_1 + h_2}$
- So integral of κ^2 varies as

$$W = \sum_e \frac{\theta^2}{(h_1 + h_2)^2}\left(|\Delta_1| + |\Delta_2|\right) \sim \sum_e \frac{\theta^2 |e|^2}{|\Delta_1| + |\Delta_2|}$$

- Edge length, triangle areas, normals are all easy to calculate
- θ needs inverse trig functions
- But θ^2 behaves a lot like $1-\cos(\theta/2)$ over interval $[-\pi,\pi]$ [draw picture]