Notes

- I'm now in X663
  - Well, sort of…
- Questions about assignment 3?

Smoothed Particle Estimate

- Take the "raw" mass estimate to get density:

$$\langle \rho(x) \rangle = \sum_j m_j W\left(x - x_j\right)$$

- Evaluate this at particles, use that to approximately normalize:

$$\langle q(x) \rangle = \sum_j q_j \frac{m_j W\left(x - x_j\right)}{\rho_j}$$

Incompressible Free Surfaces

- Actually, I lied
  - That is, regular SPH uses the previous formulation
  - For doing incompressible flow with free surface, we have a problem
  - Density drop smoothly to 0 around surface
  - This would generate huge pressure gradient, surface goes wild…
- So instead, track density for each particle as a primary variable (as well as mass!)
  - Update it with continuity equation
  - Mass stays constant however - probably equal for all particles, along with radius

Continuity equation

- Recall the equation is

$$\rho_t + u \cdot \nabla\rho = -\rho\nabla \cdot u$$

- We'll handle advection by moving particles around
- So we need to figure out right-hand side
- Divergence of velocity for one particle is

$$\nabla \cdot v = \nabla \cdot \left(v_j W\left(x - x_j\right)\right) = v_j \cdot \nabla W_j$$

- Multiply by density, get SPH estimate:

$$\langle \rho\nabla \cdot v \rangle_i = \sum_j m_j v_j \cdot \nabla_i W_{ij}$$

Momentum equation

- Without viscosity: $u_t + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + g$
- Handle advection by moving particles
- Acceleration due to gravity is trivial
- Left with pressure gradient
- Naïve approach - just take SPH estimate as before

$$\frac{dv_i}{dt} = \left\langle -\frac{1}{\rho}\nabla p \right\rangle = -\sum_j m_j \frac{p_j}{\rho_j^2}\nabla_i W_{ij}$$

Conservation of momentum

- Remember momentum equation really came out of F=ma (but we divided by density to get acceleration)
- Previous slide - momentum is not conserved
  - Forces between two particles is not equal and opposite
- We need to symmetrize this somehow

$$\frac{dv_i}{dt} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla_i W_{ij}$$

SPH advection

- Simple approach: just move each particle according to its velocity
- More sophisticated: use some kind of SPH estimate of v
  - keep nearby particles moving together
  - Note: SPH estimates only accurate when particles well organized, so this is needed for complex flows
- XSPH $$\frac{dx_i}{dt} = v_i + \sum_j \frac{m_j\left(v_j - v_i\right)}{\frac{1}{2}\left(\rho_i + \rho_j\right)} W_{ij}$$

Equation of state

- Some debate - maybe need a somewhat different equation of state if free-surface involved
- E.g. [Monaghan'94]

$$p = B\left(\left(\frac{\rho}{\rho_0}\right)^7 - 1\right)$$

- For small variations, looks like gradient is the same [linearize]
  - But SPH doesn't estimate -1 exactly, so you do get different results…

Incompressible SPH

- Can actually do a pressure solve instead of using artificial compressibility
- But if we do exact projection get the same kinds of instability as collocated grids
  - And no alternative like staggered grids available
- Instead use approximate pressure solve
  - And rely on smoothing in SPH to avoid high-frequency compression waves
  - [Cummins & Rudman '99]

Fundamental Problems

- SPH smears sharp features out
  - Need lots of particles to resolve reasonable well
  - But SPH is considerably more expensive per particle than grid methods are per grid cell
- SPH surface is bumpy
  - Same issue as using marker particles

Fire

Fire

- See Nguyen, Fedkiw, Jensen SIGGRAPH'02
- Gaseous fuel/air mix (from a burner, or a hot piece of wood, or …) heats up
- When it reaches ignition temperature, starts to burn
  - "blue core" - see the actual flame front due to emission lines of excited hydrocarbons
- Gets really hot while burning - glows orange from blackbody radiation of smoke/soot
- Cools due to radiation, mixing
  - Left with regular smoke

Defining the flow

- Inside and outside blue core, regular incompressible flow with buoyancy
- But an interesting boundary condition at the flame front
  - Gaseous fuel and air chemically reacts to produce a different gas with a different density
  - Mass is conserved, so volume has to change
  - Gas instantly expands at the flame front
- And the flame front is moving too
  - At the speed of the flow plus the reaction speed

Interface speed

- Interface = flame front = blue core surface
- $D=V_f-S$ is the speed of the flame front
  - It moves with the fuel flow, and on top of that, moves according to reaction speed S
  - S is fixed for a given fuel mix
- We can track the flame front with a level set ϕ
- Level set moves by

$$\phi_t + D|\nabla\phi| = 0$$
$$\phi_t + u_{LS} \cdot \nabla\phi = 0$$

- Here u_{LS} is u_f-Sn

Numerical method

- For water we had to work hard to move interface accurately
- Here it's ok just to use semi-Lagrangian method (with reinitialization)
- Why?
  - We're not conserving volume of blue core - if reaction is a little too fast or slow, that's fine
  - Numerical error looks like mean curvature
  - Real physics actually says reaction speed varies with mean curvature! (burn rate connected with surface area)

Conservation of mass

- Mass per unit area entering flame front is $\rho_f(V_f-D)$ where
  - $V_f=u_f \cdot n$ is the normal component of fuel velocity
  - D is the (normal) speed of the interface
- Mass per unit area leaving flame front is $\rho_h(V_h-D)$ where
  - $V_h=u_h \cdot n$ is the normal component of hot gaseous products velocity
- Equating the two gives:

$$\rho_f\left(V_f - D\right) = \rho_h\left(V_h - D\right)$$

Velocity jump

- Plugging interface speed D into conservation of mass at the flame front gives:

$$\rho_f S = \rho_h\left(V_h - V_f + S\right)$$
$$\rho_h V_h = \rho_h V_f + \rho_f S - \rho_h S$$
$$V_h = V_f + \left(\frac{\rho_f}{\rho_h} - 1\right)S$$

Ghost velocities

- This is a "jump condition": how the normal component of velocity jumps when you go over the flame interface
- This lets us define a "ghost" velocity field that is continuous
  - When we want to get a reasonable value of u_h for semi-Lagrangian advection of hot gaseous products on the fuel side of the interface, or vice versa (and also for moving interface)
  - When we compute divergence of velocity field
- Simply take the velocity field, add/subtract (ρ_f/ρ_h-1)Sn

Conservation of momentum

- Momentum is also conserved at the interface
- Fuel momentum per unit area "entering" the interface is
$$\rho_f V_f (V_f - D) + p_f$$
- Hot gaseous product momentum per unit area "leaving" the interface is
$$\rho_h V_h (V_h - D) + p_h$$
- Equating the two gives
$$\rho_f V_f (V_f - D) + p_f = \rho_h V_h (V_h - D) + p_h$$

Simplifying

- Make the equation look nicer by taking conservation of mass:
$$\rho_f (V_f - D) = \rho_h (V_h - D)$$
multiplying both sides by -D:
$$\rho_f (-D)(V_f - D) = \rho_h (-D)(V_h - D)$$
and adding to previous slide's equation:
$$\rho_f (V_f - D)^2 + p_f = \rho_h (V_h - D)^2 + p_h$$

Pressure jump

- This gives us jump in pressure from one side of the interface to the other
- By adding/subtracting the jump, we can get a reasonable continuous extension of pressure from one side to the other
  - For taking the gradient of p to make the flow incompressible after advection
- Note when we solve the Poisson equation density is NOT constant, and we have to incorporate jump in p (known) just like we use it in the pressure gradient

Temperature

- We don't want to get into complex (!) chemistry of combustion
- Instead just specify a time curve for the temperature
  - Temperature known at flame front ($T_{ignition}$)
  - Temperature of a chunk of hot gaseous product rises at a given rate to T_{max} after it's created
  - Then cools due to radiation

Temperature cont'd

- For small flames (e.g. candles) can model initial temperature rise by tracking time since reaction: $Y_t + u \cdot \nabla Y = 1$ and making T a function of Y
- For large flames ignore rise, just start flame at T_{max} (since transition region is very thin, close to blue core)
- Radiative cooling afterwards:
$$T_t + u \cdot \nabla T = -c_T \left(\frac{T - T_{air}}{T_{max} - T_{air}} \right)^4$$

Smoke concentration

- Can do the same as for temperature: initially make it a function of time Y since reaction (rising from zero)
  - And ignore this regime for large flames
- Then just advect without change, like before
- Note: both temperature and smoke concentration play back into velocity equation (buoyancy force)

Note on fuel

- We assumed fuel mix is magically being injected into scene
  - Just fine for e.g. gas burners
  - Reasonable for slow-burning stuff (like thick wood)
- What about fast-burning material?
  - Can specify another reaction speed S_{fuel} for how fast solid/liquid fuel turned into flammable gas (dependent on temperature)
  - Track level set of solid/liquid fuel just like we did the blue core