Poisson Ratio

- Real materials are essentially incompressible (for large deformation - neglecting foams and other weird composites…)
- For small deformation, materials are usually somewhat incompressible
- Imagine stretching block in one direction
  - Measure the contraction in the perpendicular directions
  - Ratio is \( \nu \), Poisson’s ratio
- [draw experiment; \( \nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} \) ]

Putting it together

\[
E\varepsilon_{11} = \sigma_{11} - \nu \sigma_{22} - \nu \sigma_{33} \\
E\varepsilon_{22} = -\nu \sigma_{11} + \sigma_{22} - \nu \sigma_{33} \\
E\varepsilon_{33} = -\nu \sigma_{11} - \nu \sigma_{22} + \sigma_{33}
\]

- Can invert this to get normal stress, but what about shear stress?
  - Diagonalization…
- When the dust settles,
  \[
  E\varepsilon_{ij} = (1 + \nu)\sigma_{ij} \quad i \neq j
  \]

What is Poisson’s ratio?

- Has to be between -1 and 0.5
- 0.5 is exactly incompressible
  - [derive]
- Negative is weird, but possible [origami]
- Rubber: close to 0.5
- Steel: more like 0.33
- Metals: usually 0.25-0.35
- What should cork be?
**Inverting…**

\[
\sigma = E \left( \frac{1}{1 + \nu} I + \frac{\nu}{(1 + \nu)(1 - 2\nu)} \mathbf{1} \otimes \mathbf{1} \right) \varepsilon
\]

- For convenience, relabel these expressions
  - \( \lambda \) and \( \mu \) are called the Lamé coefficients
  - [incompressibility] \[
  \lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \\
  \mu = \frac{E}{2(1 + \nu)}
  \]
  \[
  \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2 \mu \varepsilon_{ij}
  \]

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**Linear elasticity**

- Putting it together and assuming constant coefficients, simplifies to
  \[
  \rho \dot{\varepsilon} = f_{\text{body}} + \lambda \nabla \varepsilon_{kk} + 2 \mu \nabla \cdot \varepsilon \\
  = f_{\text{body}} + \lambda \nabla \cdot \nabla x + \mu (\nabla \cdot \nabla x + \nabla \nabla \cdot x)
  \]

- A PDE!
  - We’ll talk about solving it later

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**Rayleigh damping**

- We’ll need to look at strain rate
  - How fast object is deforming
  - We want a damping force that resists change in deformation
- Just the time derivative of strain
- For Rayleigh damping of linear elasticity

\[
\sigma_{damp}^{ij} = \phi \dot{\varepsilon}_{kk} \delta_{ij} + 2 \psi \dot{\varepsilon}_{ij}
\]

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**Problems**

- Linear elasticity is very nice for small deformation
  - Linear form means lots of tricks allowed for speed-up, simpler to code, …
- But it’s useless for large deformation, or even zero deformation but large rotation
  - (without hacks)
  - Cauchy strain’s simplification sees large rotation as deformation…
- Thus we need to go back to Green strain
(Almost) Linear Elasticity

- Use the same constitutive model as before, but with Green strain tensor
- This is the simplest general-purpose elasticity model
- Animation probably doesn’t need anything more complicated
  - Except perhaps for dealing with incompressible materials

2D Elasticity

- Let’s simplify life before starting numerical methods
- The world isn’t 2D of course, but want to track only deformation in the plane
- Have to model why
  - Plane strain: very thick material, $\varepsilon_3=0$ [explain, derive $\sigma_3$]
  - Plane stress: very thin material, $\sigma_3=0$ [explain, derive $\varepsilon_3$ and new law, note change in incompressibility singularity]

Finite Volume Method

- Simplest approach: finite volumes
  - We picked arbitrary control volumes before
  - Now pick fractions of triangles from a triangle mesh
    - Split each triangle into 3 parts, one for each corner
    - E.g. Voronoi regions
    - Be consistent with mass!
  - Assume A is constant in each triangle (piecewise linear deformation)
  - [work out]
  - Note that exact choice of control volumes not critical - constant times normal integrates to zero

Finite Element Method

- #1 most popular method for elasticity problems (and many others too)
- FEM originally began with simple idea:
  - Can solve idealized problems (e.g. that strain is constant over a triangle)
  - Call one of these problems an element
  - Can stick together elements to get better approximation
- Since then has evolved into a rigorous mathematical algorithm, a general purpose black-box method
  - Well, almost black-box…
Modern Approach

- Galerkin framework (the most common)
- Find vector space of functions that solution (e.g. X(p)) lives in
  - E.g. bounded weak 1st derivative: $H^1$
- Say the PDE is $L[X] = 0$ everywhere (“strong”)
- The “weak” statement is $\int Y(p) L[X(p)] dp = 0$ for every $Y$ in vector space
- Issue: $L$ might involve second derivatives
  - E.g. one for strain, then one for div sigma
  - So $L$, and the strong form, difficult to define for $H^1$
- Integration by parts saves the day

Weak Momentum Equation

- Ignore time derivatives - treat acceleration as an independent quantity
- We discretize space first, then use “method of lines”: plug in any time integrator

$$L[X] = \rho \ddot{X} - f_{\text{body}} - \nabla \cdot \sigma$$

$$\int_{\Omega} Y L[X] = \int_{\Omega} Y (\rho \ddot{X} - f_{\text{body}} - \nabla \cdot \sigma)$$

$$= \int_{\Omega} Y \rho \ddot{X} - \int_{\Omega} Y f_{\text{body}} - \int_{\Omega} \nabla \cdot \sigma$$

$$= \int_{\Omega} Y \rho \ddot{X} - \int_{\Omega} Y f_{\text{body}} + \int_{\Omega} \sigma \nabla Y$$

Making it finite

- The Galerkin FEM just takes the weak equation, and restricts the vector space to a finite-dimensional one
  - E.g. Continuous piecewise linear - constant gradient over each triangle in mesh, just like we used for Finite Volume Method
- This means instead of infinitely many test functions $Y$ to consider, we only need to check a finite basis
- The method is defined by the basis
  - Very general: plug in whatever you want - polynomials, splines, wavelets, RBF’s, …

Linear Triangle Elements

- Simplest choice
- Take basis $\{\phi_i\}$ where $\phi_i(p) = 1$ at $p_i$ and 0 at all the other $p_j$’s
  - It’s a “hat” function
  - Then $X(p) = \sum \phi_i X(p)$ is the continuous piecewise linear function that interpolates particle positions
- Similarly interpolate velocity and acceleration
- Plug this choice of $X$ and an arbitrary $Y = \phi_j$ (for any $j$) into the weak form of the equation
- Get a system of equations (3 eq. for each $j$)
The equations

\[
\int_{\Omega} \phi_j \sum_i \rho \dddot{x}_i \phi_i - \int_{\Omega} \phi_j f_{\text{body}} + \int_{\Omega} \sigma \nabla \phi_j = 0
\]

\[
\sum_i \int_{\Omega} \rho \phi_j \phi_i \dddot{x}_i = \int_{\Omega} \phi_j f_{\text{body}} - \int_{\Omega} \sigma \nabla \phi_j
\]

• Note that \( \phi_j \) is zero on all but the triangles surrounding \( j \), so integrals simplify
• Also: naturally split integration into separate triangles