Notes

Poisson Ratio

- Real materials are essentially incompressible (for large deformation - neglecting foams and other weird composites…)
- For small deformation, materials are usually somewhat incompressible
- Imagine stretching block in one direction
  - Measure the contraction in the perpendicular directions
  - Ratio is ν, Poisson's ratio
- [draw experiment; $\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$]

What is Poisson's ratio?

- Has to be between -1 and 0.5
- 0.5 is exactly incompressible
  - [derive]
- Negative is weird, but possible [origami]
- Rubber: close to 0.5
- Steel: more like 0.33
- Metals: usually 0.25-0.35
- What should cork be?

Putting it together

$$E\varepsilon_{11} = \sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}$$
$$E\varepsilon_{22} = -\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}$$
$$E\varepsilon_{33} = -\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}$$

- Can invert this to get normal stress, but what about shear stress?
  - Diagonalization…
- When the dust settles,

$$E\varepsilon_{ij} = (1+\nu)\sigma_{ij} \quad i \neq j$$

Inverting…

$$\sigma = E\left(\frac{1}{1+\nu} I + \frac{\nu}{(1+\nu)(1-2\nu)} 1 \otimes 1\right)\varepsilon$$

- For convenience, relabel these expressions
  - λ and μ are called the Lamé coefficients
  - [incompressibility]

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Linear elasticity

- Putting it together and assuming constant coefficients, simplifies to

$$\begin{aligned}\rho\dot{v} &= f_{body} + \lambda\nabla\varepsilon_{kk} + 2\mu\nabla\cdot\varepsilon \\ &= f_{body} + \lambda\nabla\cdot\nabla x + \mu(\nabla\cdot\nabla x + \nabla\nabla\cdot x)\end{aligned}$$

- A PDE!
  - We'll talk about solving it later

Rayleigh damping

- We'll need to look at strain rate
  - How fast object is deforming
  - We want a damping force that resists change in deformation
- Just the time derivative of strain
- For Rayleigh damping of linear elasticity

$$\sigma_{ij}^{damp} = \phi\dot{\varepsilon}_{kk}\delta_{ij} + 2\psi\dot{\varepsilon}_{ij}$$

Problems

- Linear elasticity is very nice for small deformation
  - Linear form means lots of tricks allowed for speed-up, simpler to code, …
- But it's useless for large deformation, or even zero deformation but large rotation
  - (without hacks)
  - Cauchy strain's simplification sees large rotation as deformation…
- Thus we need to go back to Green strain

(Almost) Linear Elasticity

- Use the same constitutive model as before, but with Green strain tensor
- This is the simplest general-purpose elasticity model
- Animation probably doesn't need anything more complicated
  - Except perhaps for dealing with incompressible materials

2D Elasticity

- Let's simplify life before starting numerical methods
- The world isn't 2D of course, but want to track only deformation in the plane
- Have to model why
  - Plane strain: very thick material, $\varepsilon_{3\cdot}=0$ [explain, derive $\sigma_{3\cdot}$]
  - Plane stress: very thin material, $\sigma_{3\cdot}=0$ [explain, derive $\varepsilon_{3\cdot}$ and new law, note change in incompressibility singularity]

Finite Volume Method

- Simplest approach: finite volumes
  - We picked arbitrary control volumes before
  - Now pick fractions of triangles from a triangle mesh
    - Split each triangle into 3 parts, one for each corner
    - E.g. Voronoi regions
    - Be consistent with mass!
  - Assume A is constant in each triangle (piecewise linear deformation)
  - [work out]
  - Note that exact choice of control volumes not critical - constant times normal integrates to zero

Finite Element Method

- #1 most popular method for elasticity problems (and many others too)
- FEM originally began with simple idea:
  - Can solve idealized problems (e.g. that strain is constant over a triangle)
  - Call one of these problems an element
  - Can stick together elements to get better approximation
- Since then has evolved into a rigourous mathematical algorithm, a general purpose black-box method
  - Well, almost black-box…

Modern Approach

- Galerkin framework (the most common)
- Find vector space of functions that solution (e.g. X(p)) lives in
  - E.g. bounded weak 1st derivative: H^1
- Say the PDE is L[X]=0 everywhere ("strong")
- The "weak" statement is ∫ Y(p)L[X(p)]dp=0 for every Y in vector space
- Issue: L might involve second derivatives
  - E.g. one for strain, then one for div sigma
  - So L, and the strong form, difficult to define for H^1
- Integration by parts saves the day

Weak Momentum Equation

- Ignore time derivatives - treat acceleration as an independent quantity
  - We discretize space first, then use "method of lines": plug in any time integrator

$$L[X] = \rho\ddot{X} - f_{body} - \nabla\cdot\sigma$$

$$\begin{aligned}\int_\Omega Y L[X] &= \int_\Omega Y\left(\rho\ddot{X} - f_{body} - \nabla\cdot\sigma\right)\\ &= \int_\Omega Y\rho\ddot{X} - \int_\Omega Y f_{body} - \int_\Omega Y\nabla\cdot\sigma\\ &= \int_\Omega Y\rho\ddot{X} - \int_\Omega Y f_{body} + \int_\Omega \sigma\nabla Y\end{aligned}$$

Making it finite

- The Galerkin FEM just takes the weak equation, and restricts the vector space to a finite-dimensional one
  - E.g. Continuous piecewise linear - constant gradient over each triangle in mesh, just like we used for Finite Volume Method
- This means instead of infinitely many test functions Y to consider, we only need to check a finite basis
- The method is defined by the basis
  - Very general: plug in whatever you want - polynomials, splines, wavelets, RBF's, …

Linear Triangle Elements

- Simplest choice
- Take basis $\{\phi_i\}$ where $\phi_i(p)=1$ at p_i and 0 at all the other p_j's
  - It's a "hat" function
- Then $X(p)=\sum_i x_i\phi_i(p)$ is the continuous piecewise linear function that interpolates particle positions
- Similarly interpolate velocity and acceleration
- Plug this choice of X and an arbitrary $Y=\phi_j$ (for any j) into the weak form of the equation
- Get a system of equations (3 eq. for each j)

The equations

$$\int_{\Omega} \phi_j \sum_i \rho \ddot{x}_i \phi_i - \int_{\Omega} \phi_j f_{body} + \int_{\Omega} \sigma \nabla \phi_j = 0$$

$$\sum_i \int_{\Omega} \rho \phi_j \phi_i \ddot{x}_i = \int_{\Omega} \phi_j f_{body} - \int_{\Omega} \sigma \nabla \phi_j$$

- Note that ϕ_j is zero on all but the triangles surrounding j, so integrals simplify
- Also: naturally split integration into separate triangles