Notes

- Please read O'Brien, Bargteil and Hodgins, "Graphical modeling and animation of ductile fracture", SIGGRAPH'02

Plasticity

- Recall we split the current strain into
  - an elastic part (will vanish when applied forces removed and system comes to rest)
  - and a plastic part (permanent)
- Stress is computed just from elastic strain (and its rate of change)
- We need rules for **when** plastic strain changes, and **how fast**
  - In multiple dimensions this isn't trivial

Yield criteria

- Lots of complicated stuff happens when materials yield
  - Metals: dislocations moving around
  - Polymers: molecules sliding against each other
  - Etc.
- Difficult to characterize exactly when plasticity (yielding) starts
  - Work hardening etc. mean it changes all the time too
- Approximations needed
  - Big two: Tresca and Von Mises

Yielding

- First note that shear stress is the important quantity
  - Materials (almost) never can permanently change their volume
  - Plasticity should ignore volume-changing stress
- So make sure that if we add kI to σ it doesn't change yield condition

Tresca yield criterion

- This is the simplest description:
  - Change basis to diagonalize σ
  - Look at normal stresses (i.e. the eigenvalues of σ)
  - No yield if $\sigma_{max}-\sigma_{min} \leq \sigma_Y$
- Tends to be conservative (rarely predicts yielding when it shouldn't happen)
- But, not so accurate for some stress states
  - Doesn't depend on middle normal stress at all
- Big problem (mathematically): not smooth

Von Mises yield criterion

- If the stress has been diagonalized:

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \leq \sigma_Y$$

- More generally: $\sqrt{\tfrac{3}{2}}\sqrt{\|\sigma\|_F^2-\tfrac{1}{3}Tr(\sigma)^2} \leq \sigma_Y$
- This is the same thing as the Frobenius norm of the deviatoric part of stress
  - i.e. after subtracting off volume-changing part:

$$\sqrt{\tfrac{3}{2}}\left\|\sigma-\tfrac{1}{3}Tr(\sigma)I\right\|_F \leq \sigma_Y$$

Linear elasticity shortcut

- For linear (and isotropic) elasticity, apart from the volume-changing part which we cancel off, stress is just a scalar multiple of strain
  - (ignoring damping)
- So can evaluate von Mises with elastic strain tensor too (and an appropriately scaled yield strain)

Perfect plastic flow

- Once yield condition says so, need to start changing plastic strain
- The magnitude of the change of plastic strain should be such that we stay on the yield surface
  - I.e. maintain $f(\sigma)=0$
    (where $f(\sigma)\leq 0$ is, say, the von Mises condition)
- The direction that plastic strain changes isn't as straightforward
- "Associative" plasticity: $\dot{\varepsilon}_p = \gamma\frac{\partial f}{\partial\sigma}$

Algorithm

- After a time step, check von Mises criterion: is $f(\sigma) = \sqrt{\tfrac{3}{2}}\|dev(\sigma)\|_F - \sigma_Y > 0$?
- If so, need to update plastic strain:

$$\varepsilon_p^{new} = \varepsilon_p + \gamma \frac{\partial f}{\partial \sigma} = \varepsilon_p + \gamma \sqrt{\tfrac{3}{2}} \frac{dev(\sigma)}{\|dev(\sigma)\|_F}$$

  - with γ chosen so that $f(\sigma^{new})=0$ (easy for linear elasticity)

Work hardening

- May well not need it for graphics
- But just in case, the simplest model:
  - Change yield stress to $\sigma_{Y0}+K\alpha$ where $\alpha=0$ initially (K is the "isotropic hardening modulus")
  - Change yield von Mises yield condition to

$$\sqrt{\tfrac{3}{2}}\|dev(\sigma)-\beta\|_F - \sigma_Y \le 0$$

  - where β is the centre of the yield surface, initially 0

$$\varepsilon_p^{new} = \varepsilon_p + \gamma \frac{dev(\sigma)-\beta}{\|dev(\sigma)-\beta\|_F}$$

$$\alpha^{new} = \alpha + \gamma$$

$$\beta^{new} = \beta + \tfrac{2}{3}\gamma H \frac{dev(\sigma)-\beta}{\|dev(\sigma)-\beta\|_F}$$

Creep

- Instead of instantaneously changing plastic strain in response to changing stress, let it change in time
- Elastic strain then decays exponentially
  - To zero: Maxwell fluid
  - To some fixed lower limit: more general
- If creep is a large effect, fixed mesh Lagrangian methods are bad
- If creep is small, maybe not necessary to include in animation

Viscoelasticity

- Some materials don't really have an elastic regime
  - As soon as you apply force, creep deformation begins
- Over long time, behave like a fluid
  - No shear forces resisted
- Over short time, behave like a solid
  - Bounce elastically
- Called "viscoelastic"
  - Confused sometimes with regular elastic materials with damping (a.k.a. viscosity)
- Everyday examples:
  - Cornstarch/water
  - Silly putty

Fracture

- If no plasticity before fracture occurs, called “brittle” (otherwise, “ductile”)
- Much of engineering literature concerned with crack propagation
  - Once a fracture has started, how fast does it propagate, how much energy or force is needed to continue it, …
- For graphics just concerned with when fracture occurs, and how to implement it
  - Elastoplastic modeling handles the rest

Stress on elements

- Easiest approach: loop over elements looking at stress
  - Compare max eigenvalue of stress to tensile fracture threshhold
    (usually assume no fracture in compression)
  - Associated eigenvector should be normal to new fracture surface
- But how do we put in that fracture surface?

Fracturing elements

- There isn’t an obvious place in the element to choose the plane to go
- Generally will want it to meet up with cracks in neighbouring elements…
- Do not want to arbitrarily split element (can get slivers)
- Instead:
  - Pick element face whose normal is closest to eigenvector
  - Mark that face as separated
  - Check corners of face to see if separated, duplicate if so (splitting up mass appropriately)

Separating faces

- Related to engineering “cohesive surface elements” (where crack path is known)
- Check for separated nodes based on graph connectedness:
  - Form graph where each vertex is an incident element, edges correspond to non-separated faces
  - If the graph has more than component, node must be split, one copy for each component
  - Split the mass up according to volumes/densities of incident tets

Fracture surface

- Problem: looks terrible if underlying mesh is regular
  - Not so great even if mesh is irregular but coarse
- Can be alleviated in rendering by changing fracture surface to a fractally-roughened higher detail surface
  - See Smith, Witkin, Baraff "Fast and controllable simulation of the shattering of brittle objects", Eurographics'01

Node-based fracture

- Need a fracture criterion evaluated at nodes instead of elements
- But stress doesn't "live" there
- Simple approach:
  - use an average of stresses on surrounding elements, perform test as before
- More complex: form "separation tensor"
  - See O'Brien and Hodgins for details
  - Basic idea: split stress in each element into tensile and compressive parts (use signs of eigenvalues)
  - Get tensile and compressive forces on nodes
  - Form separation tensor from these

Introducing fracture surface

- Eigenvector gives normal to new fracture surface
- Also want fracture surface to pass through node: so begin by duplicating the node
- This will split up the neighbouring elements - need to remesh (and eliminate T-junctions)
- Need to be careful to avoid slivers: if fracture plane passes very close by another node, snap it to the node and avoid the sliver
- Redistribute mass of the original node to the two copies

Rigid shortcut

- For brittle fracture, generally don't see (or care about) deformation
  - So animate pieces as rigid bodies, but when collisions occur, evaluate internal stress to see about fractures
  - See Müller, et al., "Real-time simulation of deformation and fracture of stiff materials", 2001

Collisions

- Note that when fracture occurs, bits of material are exactly touching
- Can cause difficulties for “robust” algorithms (that assume and maintain separation between objects)
- Generally need to either artificially separate at fracture, or allow for small interpenetration

Other material effects

- Heat: any material property could be made temperature dependent
  - Need to solve auxiliary heat equation to let heat diffuse through material:
$$\frac{\partial T}{\partial t} = \nabla \cdot \left(k \nabla T \right)$$
  - Unless k is very small, best to do this with implicit methods (Backward Euler typically)
  - Use FVM (or equivalent linear FEM)
  - Conductivity k can be just a constant number (get Laplacian) or could be a SPD tensor…
  - [yield stress]
  - [thermal stress]