Notes

- Assignment #1
  - Uniform distribution in a disk:
    - Either pick points uniformly in square and discard those outside disk
    - Or take r=sqrt(rand), theta=2*pi*rand
  - Shortcut: if you implement Backward Euler and Forward Euler, just reuse them in Trapezoidal Rule

Constrained Dynamics

- Want “natural” dynamics but subject to constraint
- Last time: work with regular system, but add extra forces/impulses to satisfy constraint (at least approximately)
- Now: get rid of constraint equation altogether
  - Parameterize system so constraint automatically satisfied
  - Last time, the hard part was satisfying the constraint
  - This time, the hard part is satisfying physics!

Generalized Coordinates

- Say “positions” of system in vector x
- Constraint C(x)=0
- Find parameterization of the constraint manifold x=X(q)
  - C(X(q))=0 for all q
  - For every x with C(x)=0, there is a q s.t. x=X(q)
- The q vector is the generalized coordinates
- Example: pendulum - 6 x coordinates with 5 constraints, or 1 q gen. coordinate (angle)
- No redundancy: cannot drift, should be fast for lots of constraints

Problems Ahead

- Math can get fairly nasty if the parameterization isn’t simple
  - Many people use Maple/Mathematica/etc. to crunch the expressions, generate code
- Parameterization could have pitfalls
  - [Gimbal-lock]
  - Some degenerate redundancies (multiple values of q mapping to same x)
  - End up with ill-conditioned system (in the limit, underdetermined: more than one direction for q to evolve)

General flavour

- Just look at constraint-free reparameterization (e.g. going to spherical coordinates)
- Say x=X(q), and inverse map also is well defined: q=Q(x)

$$\dot{q} = \frac{\partial Q}{\partial x}\dot{x} \qquad \dot{x} = \frac{\partial X}{\partial q}\dot{q}$$

$$\ddot{q} = \frac{\partial Q}{\partial x}\ddot{x} + \frac{\partial^2 Q}{\partial x^2} : \dot{x}\dot{x}^T$$

Getting rid of x′s…

$$\begin{aligned}
\ddot{q} &= \frac{\partial Q}{\partial x} M^{-1} F + \frac{\partial^2 Q}{\partial x^2} : \left(\frac{\partial X}{\partial q} \dot{q}\dot{q}^T \frac{\partial X}{\partial q}^T \right) \\
&= \frac{\partial X}{\partial q}^{-1} M^{-1} F + \frac{\partial^2 Q}{\partial x^2} : \left(\frac{\partial X}{\partial q} \dot{q}\dot{q}^T \frac{\partial X}{\partial q}^T \right) \\
&= \left(\frac{\partial X}{\partial q}^T M \frac{\partial X}{\partial q} \right)^{-1} \left(\frac{\partial X}{\partial q}^{-T} F \right) + \frac{\partial^2 Q}{\partial x^2} : \left(\frac{\partial X}{\partial q} \dot{q}\dot{q}^T \frac{\partial X}{\partial q}^T \right) \\
&= \tilde{M}^{-1} \left(\tilde{F} + \tilde{F}_{inertial} \right)
\end{aligned}$$

Let′s go on

- Not so important for passive motion
  - Critical for robotics
  - Important for human animation, but often better to directly specify joint angles (active motion anyways!)
  - If joint angles scripted, still can have character translate/rotate thorugh space
    - Same as a rigid body (angular momentum is conserved) except inertia tensor in object space changes in time
  - For the occasional use of constraints in passive motion, easier to use soft constraints and/or Lagrange multipliers

Deformable Objects

- In reality, no such thing as a rigid body
- Lots of things aren't anywhere close
  - All of your body except your bones
  - Clothing, and most other thin objects
  - Damaged/fractured objects
  - Water and other fluids
  - …
- Lots of degrees of freedom
  - -> painful to animate
- The math we need: continuum mechanics

Lagrangian vs. Eulerian

- Continuum: motion of a chunk of matter depends on nearby matter
- Two ways of looking at it
- Lagrangian: (e.g. particle systems)
  - Identify chunks of matter, track their positions (and velocities, accelerations, etc.) over time
- Eulerian: (will come later)
  - Forget identities of chunks of matter, instead just focus on how matter flows through space
  - Track velocity (and material properties) at fixed points in space
- [draw it - rigid chunk example]

Examples

- Elastic object, small deformation
  - Elastic means when force is removed, will try to return to original shape
  - E.g. [solid rubber ball]
  - Lagrangian works great
  - Eulerian - might have difficulty
- Completely fluid object, large deformation
  - E.g. [coffee]
  - Lagrangian has problems
  - Eulerian - works great

Elastic objects

- Simplest model: masses and springs
- Split up object into regions
- Integrate density in each region to get mass (if things are uniform enough, perhaps equal mass)
- Connect up neighbouring regions with springs
  - Careful: need chordal graph
- Now it's just a particle system
  - When you push on a node, neighbours pulled along with it, etc.

Masses and springs

- But: how strong should the springs be? Is this good in general?
  - [anisotropic examples]
- General rule: we don't want to see the mesh in the output
  - Avoid "grid artifacts"
  - We of course will have numerical error, but let's avoid obvious patterns in the error

1D masses and springs

- Look at a homogeneous elastic rod, length 1, linear density ρ
- Parameterize by p (x(p)=p in rest state)
- Split up into intervals/springs
  - $0 = p_0 < p_1 < \ldots < p_n = 1$
  - Mass $m_i=\rho(p_{i+1}-p_{i-1})/2$ (+ special cases for ends)
  - Spring i+1/2 has rest length $L_{i+\frac{1}{2}} = p_{i+1} - p_i$

  and force $f_{i+\frac{1}{2}} = k_{i+\frac{1}{2}} \dfrac{x_{i+1} - x_i - L_{i+\frac{1}{2}}}{L_{i+\frac{1}{2}}}$

Figuring out spring constants

- So net force on i is

$$F_i = k_{i+\frac{1}{2}} \frac{x_{i+1} - x_i - L_{i+\frac{1}{2}}}{L_{i+\frac{1}{2}}} - k_{i-\frac{1}{2}} \frac{x_i - x_{i-1} - L_{i-\frac{1}{2}}}{L_{i-\frac{1}{2}}}$$
$$= k_{i+\frac{1}{2}}\left(\frac{x_{i+1} - x_i}{p_{i+1} - p_i} - 1\right) - k_{i-\frac{1}{2}}\left(\frac{x_i - x_{i-1}}{p_i - p_{i-1}} - 1\right)$$

- We want mesh-independent response (roughly), e.g. for static equilibrium
  - Rod stretched the same everywhere: $x_i=\alpha p_i$
  - Then net force on each node should be zero (add in constraint force at ends…)

Young's modulus

- So each spring should have the same k
  - Note we divided by the rest length
  - Some people don't, so they have to make their constant scale with rest length
- The constant k is a material property (doesn't depend on our discretization) called the Young's modulus
  - Often written as E
- The one-dimensional Young's modulus is simply force per percentage deformation

The continuum limit

- Imagine Δp (or Δx) going to zero
  - Eventually can represent any kind of deformation
  - [note force and mass go to zero too]

$$\ddot{x}(p) = \frac{1}{\rho}\frac{\partial}{\partial p}\left(E(p)\left(\frac{\partial}{\partial a}x(p) - 1\right)\right)$$

  - If density and Young's modulus constant,

$$\frac{\partial^2 x}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 x}{\partial p^2}$$

Sound waves

- Try solution $x(p,t)=x_0(p-ct)$
- And $x(p,t)=x_0(p+ct)$
- So speed of sound in rod is $\sqrt{\frac{E}{\rho}}$
- Courant-Friedrichs-Levy (CFL) condition:
  - Numerical methods only will work if information transmitted numerically at least as fast as in reality (here: the speed of sound)
  - Usually the same as stability limit for explicit methods [what are the eigenvalues here]
  - Implicit methods transmit information infinitely fast

Why?

- Are sound waves important?
  - Visually? Usually not
- However, since speed of sound is a material property, it can help us get to higher dimensions
- Speed of sound in terms of one spring is

$$c=\sqrt{\frac{kL}{m}}$$

- So in higher dimensions, just pick k so that c is constant
  - m is mass around spring [triangles, tets]

Damping

- Figuring out how to scale damping is more tricky
- Go to differential equation (no mesh)

$$\frac{\partial^2 x}{\partial t^2}=\frac{1}{\rho}\frac{\partial}{\partial p}\left(E\left(\frac{\partial x}{\partial p}-1\right)+D\frac{\partial v}{\partial p}\right)$$

- So spring damping should be

$$f_{i+½}=k_{i+½}\frac{x_{i+1}-x_i-L_{i+½}}{L_{i+½}}+d_{i+½}\frac{v_{i+1}-v_i}{L_{i+½}}$$

Extra effects with springs

- (Brittle) fracture
  - Whenever a spring is stretched too far, break it
  - Issue with loose ends…
- Plasticity
  - Whenever a spring is stretched too far, change the rest length part of the way

Mass-spring problems

- [anisotropy]
- [stretching, Poisson's ratio]
- [2D bending]
- More generally: implicit integration? contact/collision?