Learning Objectives

At the end of the class you should be able to:

- explain how cycle checking and multiple-path pruning can improve efficiency of search algorithms
- explain the complexity of cycle checking and multiple-path pruning for different search algorithms
- justify why the monotone restriction is useful for A^* search
- predict whether forward, backward, bidirectional or island-driven search is better for a particular problem
- demonstrate how dynamic programming works for a particular problem

Summary of Search Strategies

Strategy	Frontier Selection	Complete	Halts	Space
Depth-first	Last node added			
Breadth-first	First node added			
Best-first	Global min $h(p)$			
Lowest-cost-first	Minimal $cost(p)$			
A^*	Minimal $f(p)$			

Complete — if there a path to a goal, it can find one, even on infinite graphs.

Halts — on finite graph (perhaps with cycles).

Space — as a function of the length of current path

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Strategy	Frontier Selection	Complete	Halts	Space
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Breadth-first	First node added	Yes	No	Exp
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Lowest-cost-first	Minimal $cost(p)$	Yes	No	Exp
A^*	Minimal $f(p)$	Yes	No	Exp

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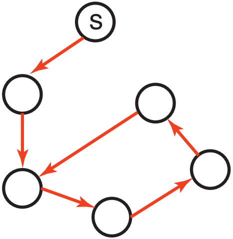
Cycle Pruning

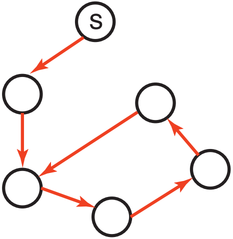


- A searcher can prune a path that ends in a node already on the path, without removing an optimal solution.

Graph searching with cycle pruning

Input: a graph,
a set of start nodes,
Boolean procedure $goal(n)$ that tests if n is a goal node.
$frontier := \{\langle s \rangle : s \text{ is a start node}\}$
while *frontier* is not empty:
  select and **remove** path $\langle n_0, \ldots, n_k \rangle$ from *frontier*
  if $n_k \notin \{n_0, \ldots, n_{k-1}\}$:
    if $goal(n_k)$:
      return $\langle n_0, \ldots, n_k \rangle$
    $Frontier := Frontier \cup \{\langle n_0, \ldots, n_k, n \rangle : \langle n_k, n \rangle \in A\}$

Cycle Pruning



- In depth-first search, checking for cycles can be done in ________ time in path length.

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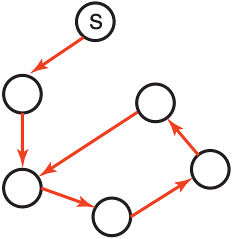

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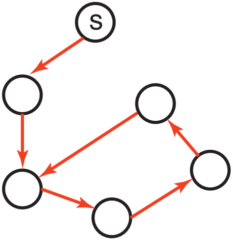

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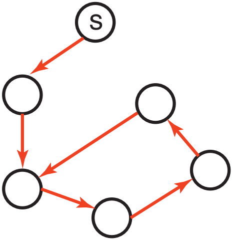

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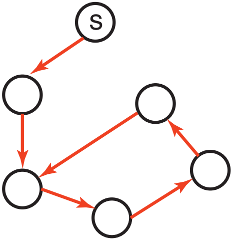
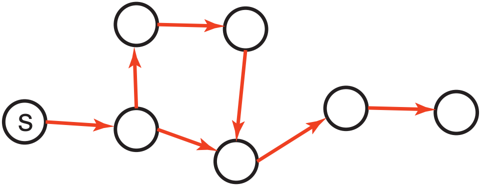


- In depth-first search, checking for cycles can be done in constant time in path length.
- For other methods, checking for cycles can be done in linear time in path length.
- With cycle pruning, which algorithms halt on finite graphs?

Multiple-Path Pruning



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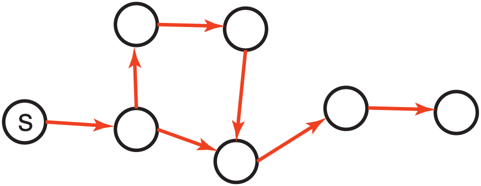

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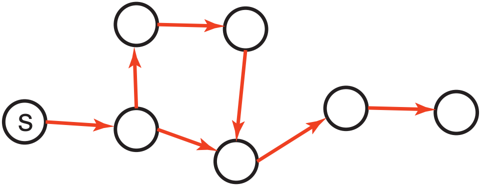

- Multiple path pruning: prune a path to node *n* that the searcher has already found a path to.
- What needs to be stored?
- Lowest-cost-first search with multiple-path pruning is Dijkstra's algorithm, and is the same as A^* with multiple-path pruning and a heuristic function of 0.

Graph searching with multiple-path pruning

Input: a graph,
a set of start nodes,
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add n_k to *expanded*
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- Which search algorithms with multiple-path pruning always halt on finite graphs?
- What is the time overhead of multiple-path pruning?
- What is the space overhead of multiple-path pruning?
- Is it better for depth-first or breadth-first searches?
- Can multiple-path pruning prevent an optimal solution being found?

Multiple-Path Pruning & Optimal Solutions

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- remove all paths from the frontier that use the longer path.
- change the initial segment of the paths on the frontier to use the lower-cost path.
- ensure this doesn't happen. Make sure that the lower-cost path to a node is expanded first.

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  We can ensure this doesn't occur if
  $h(n') - h(n) \leq cost(n', n)$.

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- If h satisfies the monotone restriction, A^* with multiple path pruning always finds a least-cost path to a goal.
- This is a strengthening of the admissibility criterion.

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- Note: when graph is dynamically constructed, the backwards graph may not be available. One might be more difficult to compute than the other.

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  - in the other direction, another method (typically depth-first) can be used to find a path to these interesting states.
  - How much is stored in the breadth-first method, can be tuned depending on the space available.

Island Driven Search

- Idea: find a set of islands between s and g.

  $$s \longrightarrow i_1 \longrightarrow i_2 \longrightarrow \ldots \longrightarrow i_{m-1} \longrightarrow g$$

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- The subproblems can be solved using islands $\Longrightarrow$ hierarchy of abstractions.

Dynamic Programming

Idea: Let $cost\_to\_goal(n)$ be the actual cost of a lowest-cost path from node n to a goal; $cost\_to\_goal(n)$ can be defined as

$$cost\_to\_goal(n) = \begin{cases} 0 & \text{if } goal(n), \\ \min_{\langle n,m\rangle \in A}(cost(\langle n, m\rangle) + cost\_to\_goal(m)) & \text{otherwise.} \end{cases}$$

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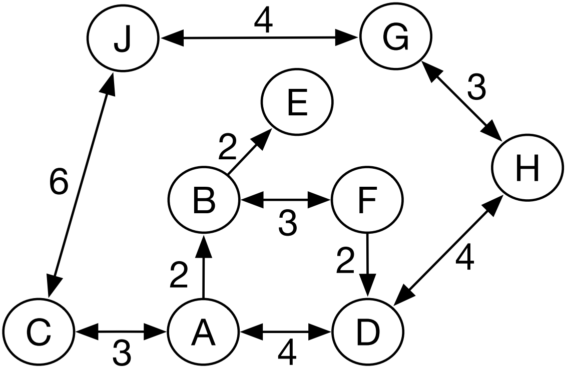
  - It requires enough space to store the graph.
  - The *cost_to_goal* function needs to be recomputed for each goal.
- Implementation detail: in Python, make *expanded* in MPP a dictionary, so *expanded*[*s*] returns the cost from *s* to goal (cost found in search).

Example graph with heuristics

Goal: G.



Example graph cost-to-goal

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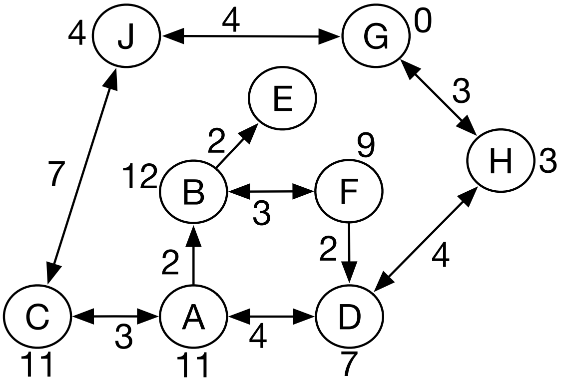

Value on nodes are *cost_to_goal* of arc.

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The heuristic function

$$h'(n) = \begin{cases} expanded[n] & \text{if } expanded[n] \text{ is defined,} \\ \max(c, h(n)) & \text{otherwise.} \end{cases}$$

is an admissible heuristic function that that satisfies the montone restriction and (generally) improves h, as it is perfect for all values less than c.