Making Decisions Under Uncertainty

What an agent should do depends on:

- The agent's ability — what options are available to it.
- The agent's beliefs — the ways the world could be, given the agent's knowledge.
  Sensing updates the agent's beliefs.
- The agent's preferences — what the agent wants and tradeoffs when there are risks.

Decision theory specifies how to trade off the desirability and probabilities of the possible outcomes for competing actions.

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- A possible world specifies a value for each decision variable and each random variable.
- For each assignment of values to *all* decision variables, there is a probability distribution over random variables.
- The probability of a proposition is undefined unless the agent conditions on the values of all decision variables.

Decision Tree for Delivery Robot

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The robot can choose to go the short way past the stairs or a long way that reduces the chance of an accident.
There is one random variable of whether there is an accident.

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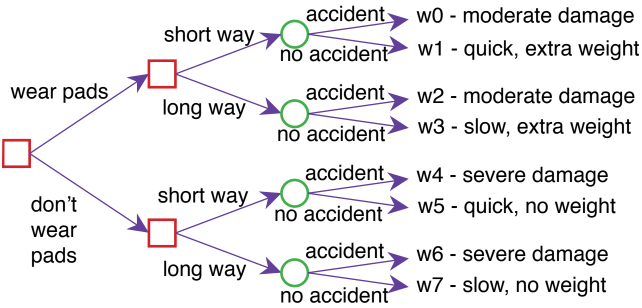

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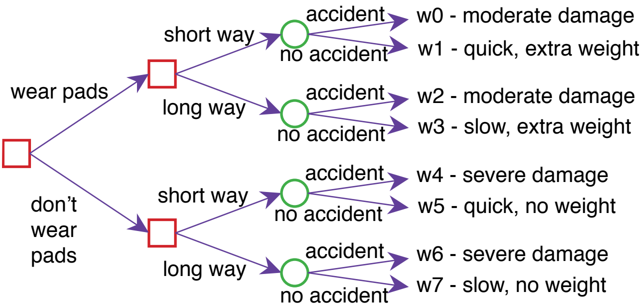

Square boxes represent decisions that the robot can make. Circles represent random variables that the robot can't observe before making its decision.

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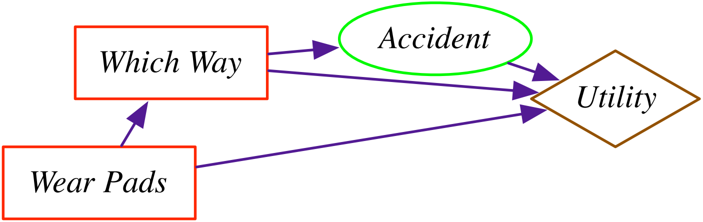
  where $u(\cdot)$ is the utility function Ω is the set of all worlds
- An optimal single decision is a decision $D = d_{max}$ whose expected utility is maximal:

$$\mathcal{E}(u \mid D = d_{max}) = \max_{d_i \in domain(D)} \mathcal{E}(u \mid D = d_i).$$

Single-stage decision networks

Extend belief networks with:

- Decision nodes that the agent chooses the value for. Domain is the set of possible actions. Drawn as rectangle.
- Utility node, whose parents are the variables on which the utility depends. Drawn as a diamond.



This shows explicitly which nodes affect whether there is an accident.

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- A conditional probability for each random variable given its parents
- (No tables associated with the decision nodes.)

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- Suppose the random variables are $X_1, \ldots, X_n$, and utility depends on $X_{i_1}, \ldots, X_{i_k}$

$$\mathcal{E}(u \mid D) \;=$$

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  To find an optimal decision:
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  - Sum out all of the random variables
  - This creates a factor on D that gives the expected utility for each value in the domain of D
  - Choose the D with the maximum value in the factor.

Example Initial Factors

Which Way	Accident	Value
long	true	0.01
long	false	0.99
short	true	0.2
short	false	0.8

Which Way	Accident	Wear Pads	Value
long	true	true	30
long	true	false	0
long	false	true	75
long	false	false	80
short	true	true	35
short	true	false	3
short	false	true	95
short	false	false	100

After summing out Accident

Which Way	Wear Pads	Value
long	true	74.55
long	false	79.2
short	true	83.0
short	false	80.6

Decision Networks

- flat or modular or hierarchical
- explicit states or features or individuals and relations
- static or finite stage or indefinite stage or infinite stage
- fully observable or partially observable
- deterministic or stochastic dynamics
- goals or complex preferences
- single agent or multiple agents
- knowledge is given or knowledge is learned
- perfect rationality or bounded rationality

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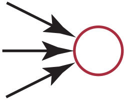
- A more typical scenario is where the agent:
  observes, acts, observes, acts, . . .
- Subsequent actions can depend on what is observed.
  What is observed depends on previous actions.
- Often the sole reason for carrying out an action is to provide information for future actions.
  For example: diagnostic tests, spying.

Sequential decision problems

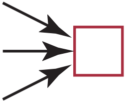
- A sequential decision problem consists of a sequence of decision variables $D_1, \ldots, D_n$.
- Each D_i has an information set of variables $parents(D_i)$, whose value will be known at the time decision D_i is made.

Decisions Networks

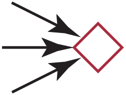
A decision network is a graphical representation of a finite sequential decision problem, with 3 types of nodes:



- A random variable is drawn as an ellipse. Arcs into the node represent probabilistic dependence.

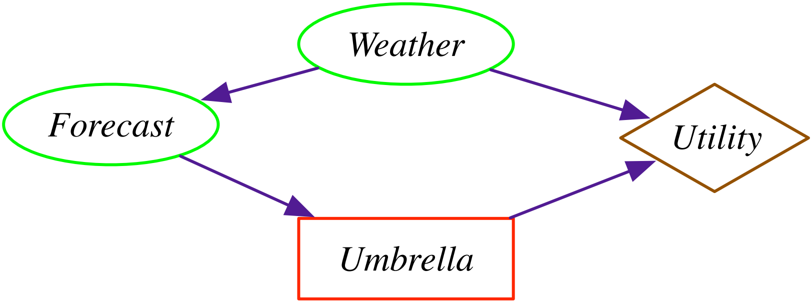


- A decision variable is drawn as an rectangle. Arcs into the node represent information available when the decision is make.



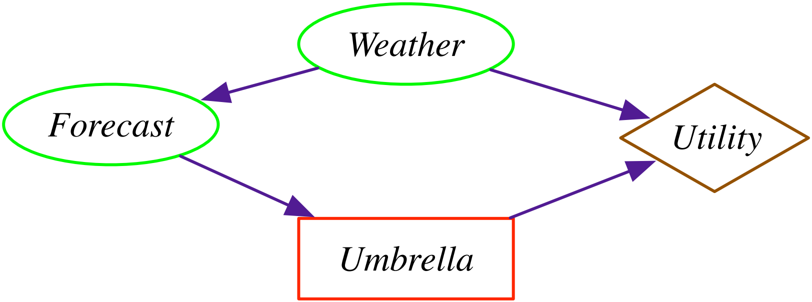
- A utility node is drawn as a diamond. Arcs into the node represent variables that the utility depends on.

Umbrella Decision Network



- The agent has to decide whether to take its umbrella.
- It observes

Umbrella Decision Network



- The agent has to decide whether to take its umbrella.
- It observes the forecast.
- It doesn't observe

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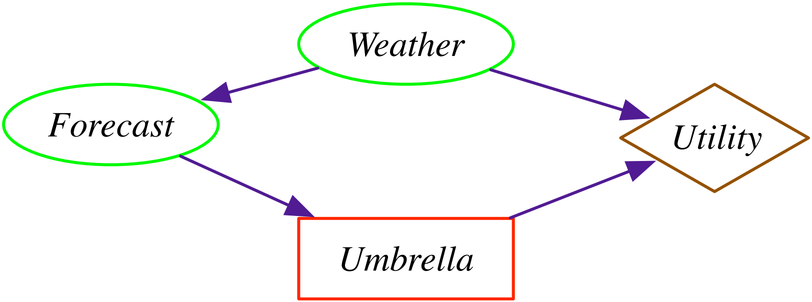

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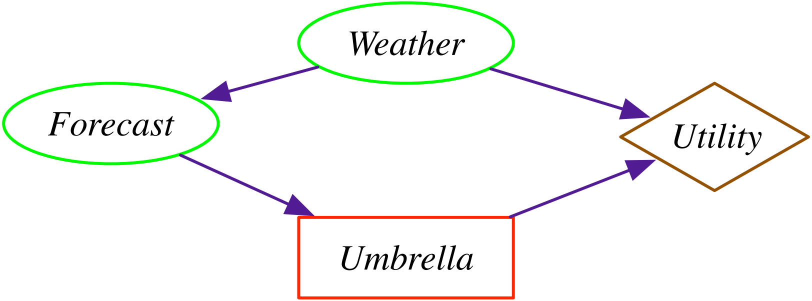

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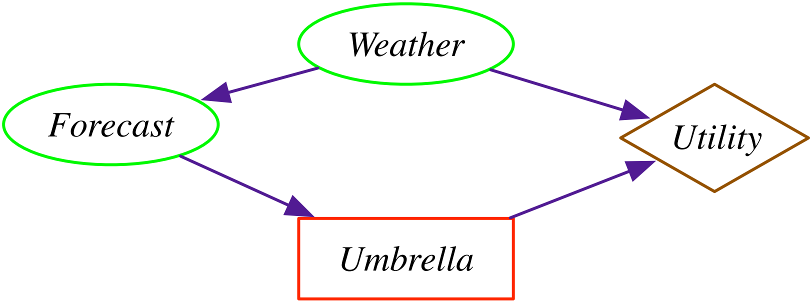

- The agent has to decide whether to take its umbrella.
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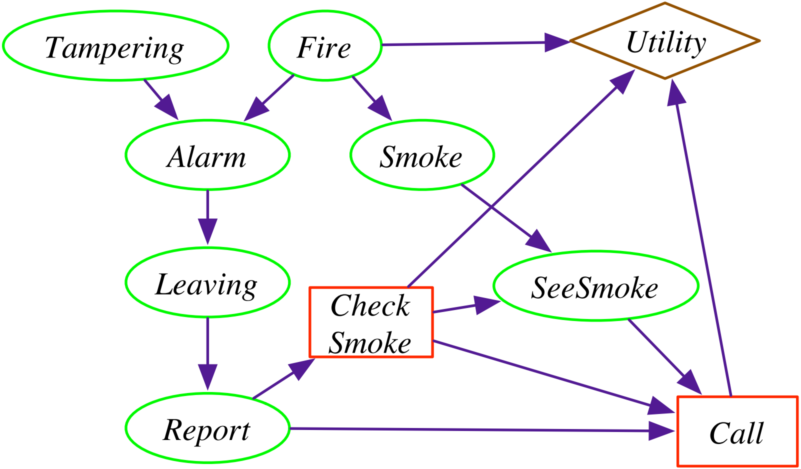
- The forecast is a noisy sensor of the weather.
- The utility depends on the weather and whether the agent takes the umbrella.

Decision Network for the Alarm Problem



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- Any parent of a decision node is a parent of subsequent decision nodes. Thus the agent remembers its previous observations.

What should an agent do?

- What an agent should do at any time depends on what it will do in the future.
- What an agent does in the future depends on what it did before.

Policies

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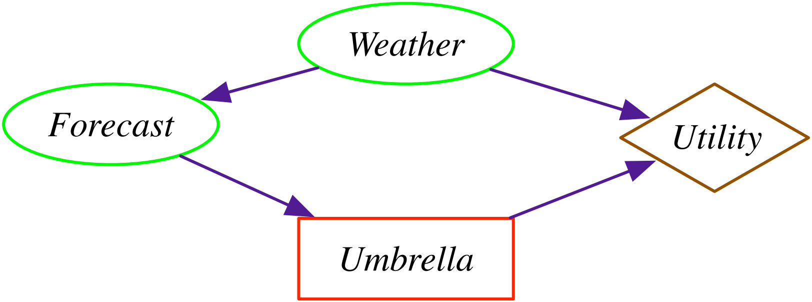
- A policy is a sequence of decision functions; one for each decision node.

Umbrella Decision Network



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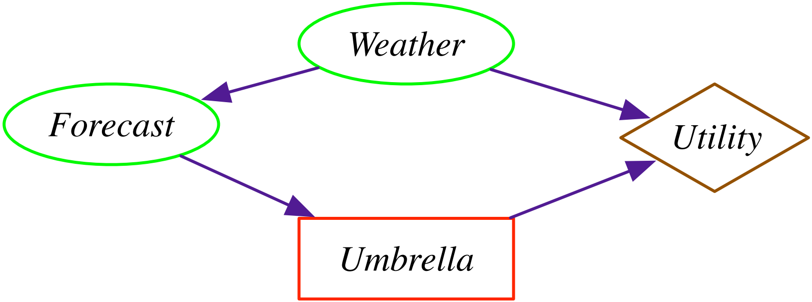
$domain(Umbbrella) = \{take, leave\}$
Some policies:

Umbrella Decision Network



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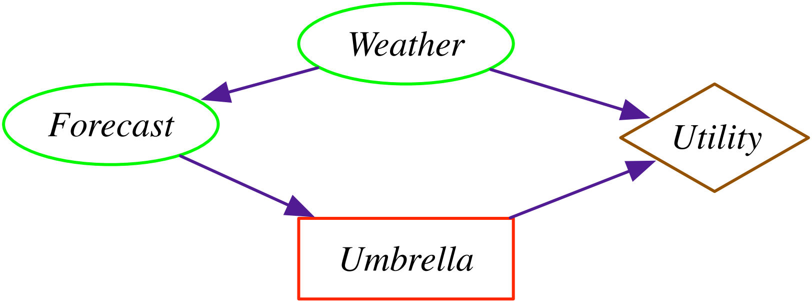

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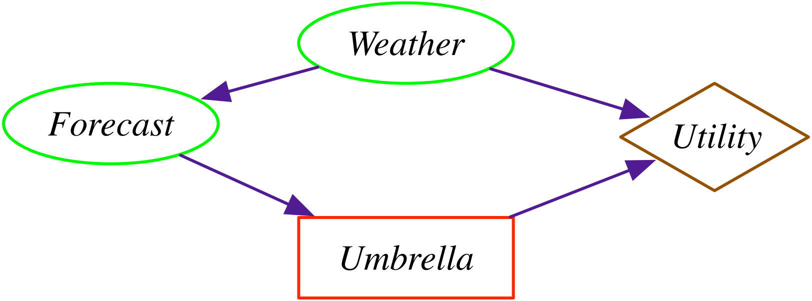

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There are $2^3 = 8$ policies

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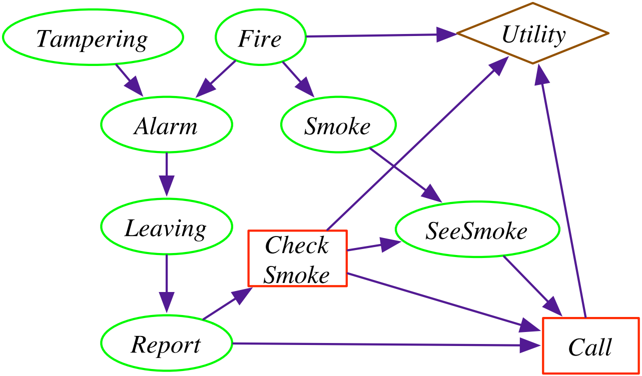

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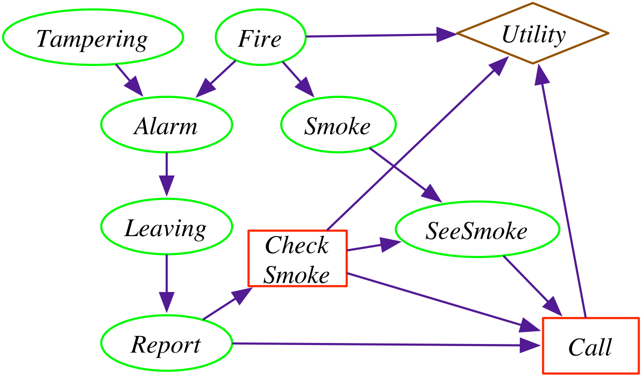

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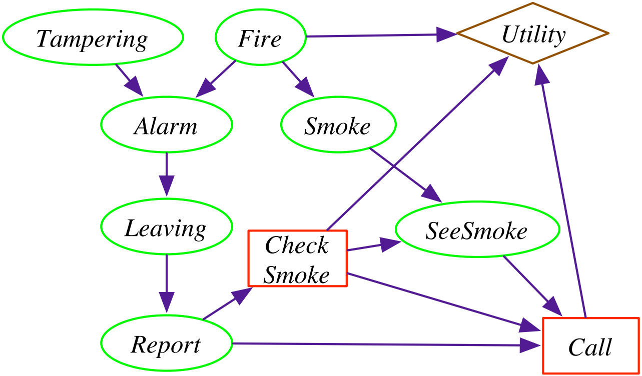

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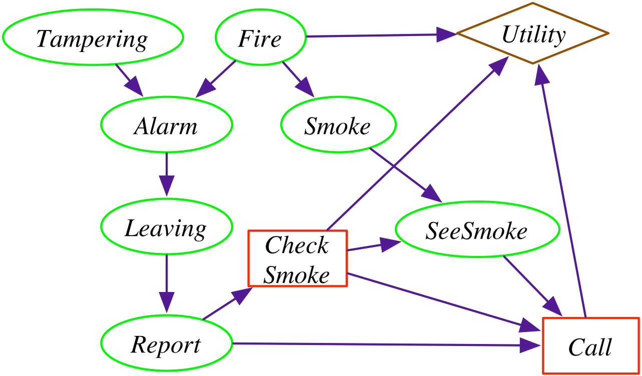

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There are $2^2 * 2^8 = 1024$ policies.

Expected Utility of a Policy

- Possible world ω satisfies policy π if ω assigns the value to each decision node that the policy specifies.
- The expected utility of policy π is

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- An optimal policy is one with the highest expected utility.

Finding an optimal policy

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- until there are no more decision nodes.
- Sum out the remaining random variables. Multiply the factors: this is the expected utility of an optimal policy.

Initial factors for the Umbrella Decision

Weather	Value
norain	0.7
rain	0.3

Weather	Fcast	Value
norain	sunny	0.7
norain	cloudy	0.2
norain	rainy	0.1
rain	sunny	0.15
rain	cloudy	0.25
rain	rainy	0.6

Weather	Umb	Value
norain	take	20
norain	leave	100
rain	take	70
rain	leave	0

Eliminating By Maximizing

f:

Fcast	Umb	Val
sunny	take	12.95
sunny	leave	49.0
cloudy	take	8.05
cloudy	leave	14.0
rainy	take	14.0
rainy	leave	7.0

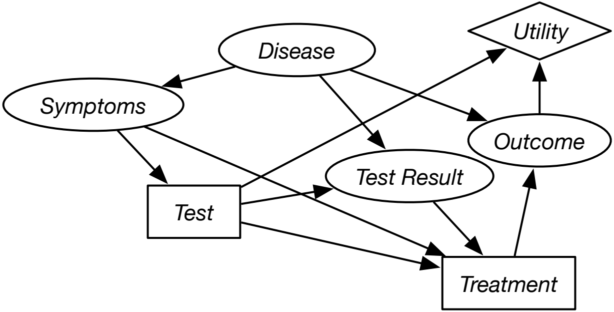
$\max_{Umb} f$:

Fcast	Val
sunny	49.0
cloudy	14.0
rainy	14.0

$\arg\max_{Umb} f$:

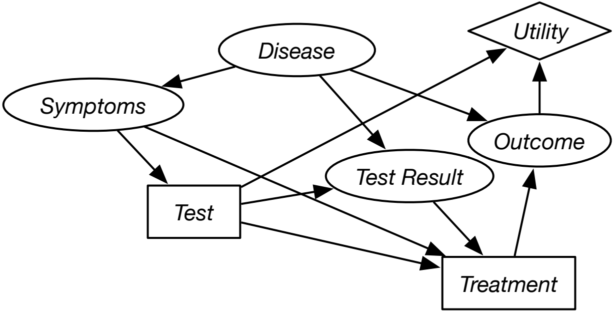
Fcast	Umb
sunny	leave
cloudy	leave
rainy	take

Exercise



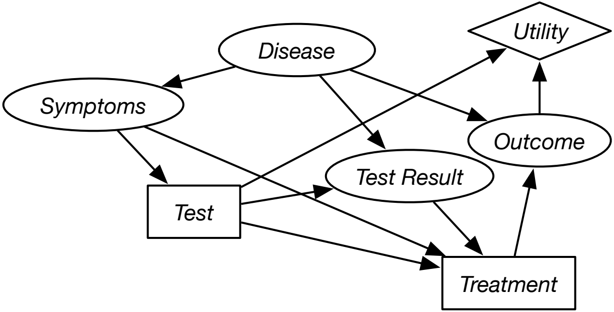
What are the factors?

Exercise



What are the factors?
Which random variables get summed out first?

Exercise



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Which random variables get summed out first?
Which decision variable is eliminated? What factor is created?

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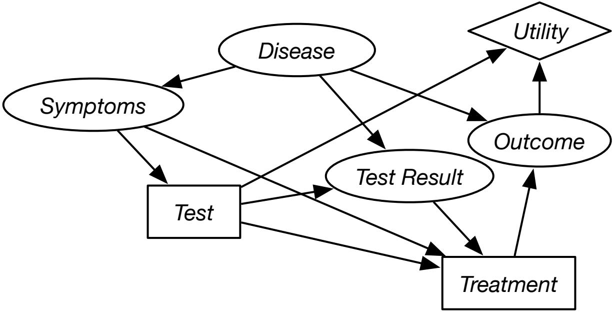

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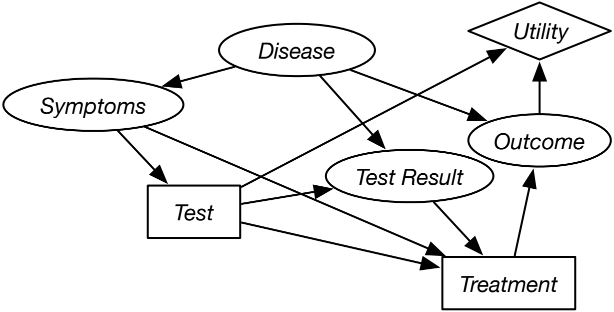

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Which decision variable is eliminated? What factor is created?
Then what is eliminated (and how)?
What factors are created after maximization?

Complexity of finding an optimal policy

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- there are assignments of values to the parents.

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If the decision variables are $D_i, \ldots, D_n$ and decision D_i has k_i binary parents and b_i possible actions:

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- optimizing in the variable elimination algorithm takes $$O\left(\sum_{i=1}^{n} b_i * 2^{k_i}\right)$$ time
- The dynamic programming algorithm is much more efficient than searching through policy space.

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- The value of information provides a bound on how much an agent should be prepared to pay for a sensor. How much is a better weather forecast worth?
- We need to be careful when adding an arc would create a cycle. E.g., how much would it be worth knowing whether the fire truck will arrive quickly when deciding whether to call them?

Value of Control

- The value of control of a variable X is the value of the network when X is a decision variable (and add no-forgetting arcs) minus the value of the network when X is a random variable.
- You need to be explicit about what information is available when you control X.
- If you control X without observing, controlling X can be worse than observing X. E.g., controlling a thermometer.
- If you keep the parents the same, the value of control is always non-negative.