An introduction to the Kadison-Singer Problem and the Paving Conjecture

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Abstract

We give a gentle introduction to the famous Kadison-Singer problem, aimed at readers without much background in functional analysis or operator theory. We explain why this problem is equivalent to some "paving" or "discrepancy" problems concerning finite-dimensional matrices.

1 Introduction

The Kadison-Singer problem was one of the central open questions in operator theory, until its recent solution by Marcus, Spielman and Srivastava [MSS13]. The importance of the problem is underscored by its connections to numerous diverse areas of mathematics. Unfortunately, unless one has a decent background in operator theory and functional analysis, the literature on this problem is hard to digest[1], and even the statement of the problem is not easily understood.

Fortunately, the Kadison-Singer problem has many equivalent forms, some of which are much easier to understand. In particular, there several problems involving finite-dimensional matrices that are equivalent to the Kadison-Singer problem, and can be easily understood by anyone with basic understanding of linear algebra.

The purpose of this document is to provide a self-contained statement of the Kadison-Singer problem and its reduction to some simple finite-dimensional problems. We only assume that the reader understands linear algebra and basic real analysis. A concise statement of the Kadison-Singer problem is:

Problem 1.1. Does every pure state on the algebra of bounded diagonal operators on (the complex Banach space) ℓ_2 have a unique extension to a state[2] on the algebra of all bounded operators on ℓ_2?

Several of the terms used in the problem statement come from functional analysis, operator theory and quantum physics, and may not be familiar to readers who have not studied those

[1] At least, the literature is hard for me to digest. Please send me any corrections or suggestions by email.
[2] As is discussed in Section 4.1, the problem is equivalent under replacing this word "state" with "pure state".

areas. The appendices of this document contain a concise list of definitions and facts that we will require from standard courses in those areas.

In their original paper, Kadison and Singer [KS59, §5] were careful not to conjecture an anwser to this question, although many authors commonly write "the Kadison-Singer conjecture" for the assertion that the answer should be "yes". The theorem of Marcus et al. shows that the answer is indeed "yes". A formal statement of their result is in Section 7.

2 A two-dimensional example

To understand the definitions, let us begin by considering a two-dimensional analog of Problem 1.1. In this case, the algebra of "bounded diagonal operators on ℓ_2" simply becomes the algebra of diagonal 2×2 matrices over the complex numbers

$$\mathbb{D}_2 \;:=\; \left\{ M = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,:\, a, d \in \mathbb{C} \right\}.$$

A linear functional on $\mathbb{D}_2$ is simply a map $f : \mathbb{D}_2 \to \mathbb{C}$ with $f(M) = f(a,d) = \alpha a + \delta d$ and $\alpha, \delta \in \mathbb{C}$. A state (see Definition C.14) is a linear function f satisfying

- $f(I) = 1$. So we must have $\delta = 1 - \alpha$.
- $f(M)$ must be real and non-negative whenever M is positive semidefinite (i.e., whenever a, d are both real and non-negative). So we must have α real and $\alpha \in [0,1]$.

A pure state (see Definition C.21) is a state f satisfying

- f cannot be written as a non-trivial convex combination of two different states. So we must have either $\alpha = 0$ or $\alpha = 1$.

So the only pure states on $\mathbb{D}_2$ are $f(M) = a$ and $f(M) = d$.

In this two-dimensional example, the algebra of "bounded operators on ℓ_2" simply becomes the algebra of two-dimensional matrices over the complex numbers

$$\mathbb{C}^{2\times 2} \;:=\; \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,:\, a, b, c, d \in \mathbb{C} \right\}.$$

A linear functional on $\mathbb{C}^{2\times 2}$ is a function $g(M) = g(a,b,c,d) = \alpha a + \beta b + \gamma c + \delta d$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Letting $G = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$, we can write $g(M) = \operatorname{tr}(GM)$. A state on $\mathbb{C}^{2\times 2}$ is a linear functional g satisfying

- $g(I) = 1$.
- $g(M)$ must be real and non-negative whenever M is Hermitian, positive semi-definite.

The first condition is equivalent to $\operatorname{tr} G = 1$. The second condition implies that $v^* G v = \operatorname{tr}(G v v^*) = g(v v^*) \geq 0$ for all $v \in \mathbb{C}^2$, which implies that G is Hermitian and positive semi-definite (see Example C.10). Conversely, any function $g(M) = \operatorname{tr}(GM)$ with G complex, positive semi-definite and $\operatorname{tr} G = 1$ is a state on $\mathbb{C}^{2\times 2}$. Thus, in this two-dimensional example, the C^*-algebraic definition of state from Definition C.14 coincides with the usual definition of state in quantum physics.

References: Watrous [Wat11, §3.1.2], Wikipedia.

A state g on $\mathbb{C}^{2\times 2}$ is an extension of a state f on $\mathbb{D}_2$ if $g(M) = f(M)$ for all $M \in \mathbb{D}_2$ (i.e., $g(a,0,0,d) = f(a,d)$). Every state f on $\mathbb{D}_2$ has a canonical extension to a state g on $\mathbb{C}^{2\times 2}$

obtained simply by defining $g(M) = g(a, b, c, d) = f(a, d)$ (since positive semi-definite matrices have non-negative diagonals). When is this extension unique?

Consider the state $f(M) = (a + d)/2$ on $\mathbb{D}_2$. This is not a pure state. Consider the linear functional $g(M) = (a + b + c + d)/2 = \operatorname{tr}(GM)$ where $G = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)/2$. This is a state on $\mathbb{C}^{2\times 2}$ since G is positive semi-definite and $\operatorname{tr} G = 1$. Clearly g is an extension of f, and $g(I) = 1$. So $g(M) = (a + d)/2$ and $g(M) = (a + b + c + d)/2$ are both states on $\mathbb{C}^{2\times 2}$ that are extensions of f. This is not a counterexample to the two-dimensional Kadison-Singer problem, as f is not a pure state.

So let us consider the pure state $f(M) = a$. Consider any linear functional $g(M) = \operatorname{tr}(GM)$ that is a state on $\mathbb{C}^{2\times 2}$ and is an extension of f. Then we must have $G = \left(\begin{smallmatrix} 1 & \beta \\ \overline{\beta} & 0 \end{smallmatrix}\right)$ for some $\beta \in \mathbb{C}$, and G positive semi-definite. (Here $\overline{\beta}$ is the complex conjugate of β.) As the diagonal entries of G are non-negative, G is positive semi-definite if and only if $\det G \geq 0$. As $\det G = -\beta\overline{\beta}$, this is non-negative only when $\beta = 0$. Thus $g(M) = a$ is the unique state on $\mathbb{C}^{2\times 2}$ that is an extension of $f(M) = a$.

Similarly, $f(M) = d$ has a unique extension to a state on $\mathbb{C}^{2\times 2}$. So we may conclude that the two-dimensional analog of the Kadison-Singer problem is true.

3 ℓ_∞ and Ultrafilters

Although the two-dimensional analog of the Kadison-Singer problem is very simple, it becomes much more interesting in infinite dimensions. The main reason is that finite-dimensional pure states have a rather trivial structure, whereas infinite-dimensional pure states are much more intricate.

To formalize this, we must begin discussing the algebra of bounded operators on ℓ_2 (denoted $B(\ell_2)$) and the algebra of bounded diagonal operators on ℓ_2 (denoted $\mathbb{D}(\ell_2)$). For the definitions of these objects, see Example B.9, Definition B.13 and Example B.14. Perhaps the most difficult task in understanding the statement of Problem 1.1 is to understand the structure of $\mathbb{D}(\ell_2)$. As explained in Example B.14, $\mathbb{D}(\ell_2)$ is isomorphic to ℓ_∞, so we will need to study ℓ_∞ in some detail. Although the ℓ_p spaces are easily understood for $p \in [1, \infty)$, the space ℓ_∞ is quite different.[3]

The key to understanding ℓ_∞ is to understand ultrafilters, which we introduce in this section. The main result is Claim 3.23, which shows that ℓ_∞ is isometrically isomorphic to the space of all continuous functions on the ultrafilters of $\mathbb{N}$.

3.1 Ultrafilters

The material in this section is primarily obtained from Hindman's survey [Hin96], the book of Hindman and Strauss [HS98, Chapter 3], and Tricki.

Definition 3.1. Let X be any non-empty set. A ***filter*** on the set X is a collection $\mathcal{F} \subset 2^X$ satisfying the following properties:

- $X \in \mathcal{F}$,
- $\emptyset \notin \mathcal{F}$,

[3] For example, whereas ℓ_p is separable for $p \in [1, \infty)$, Claim B.12 shows that ℓ_∞ is not separable. Consequently, ℓ_∞ does not have a (Schauder) basis, whereas ℓ_p does for every $p \in [1, \infty)$.

- if $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$, and
- if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

References: Hindman [HS98, Def. 2.1], Wikipedia.

Example 3.2. Let X be a topological space. Fix $x \in X$. Let $\mathcal{F}$ consist of all neighborhoods of x. Then $\mathcal{F}$ is a filter. ■

Definition 3.3. A filter $\mathcal{U} \subset 2^X$ is called an ***ultrafilter*** if

- for every $A \subseteq X$, exactly one of A or A^c $(= X \setminus A)$ is in $\mathcal{U}$.

Example 3.4. Fix any element $x \in X$. Let $\mathcal{U}_x = \{ A \subseteq X : x \in A \}$. It is easy to check that this is an ultrafilter. The ultrafilters of this form are called ***principal ultrafilters***, ***trivial ultrafilters***, or ***fixed ultrafilters***. ■

Example 3.5. Let $\mathcal{U} = \{ A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite} \}$. This is a filter, called the ***cofinite filter*** (or Fréchet filter). It is not an ultrafilter (e.g., neither the set of odd numbers nor the set of even numbers belongs to $\mathcal{U}$). ■

If X is a finite set then every ultrafilter is principal. (This follows from Claim 3.8 below). If X is infinite then, assuming the axiom of choice, there exist other ultrafilters (known as ***free ultrafilters***, or ***non-principal ultrafilters***). The axiom of choice is necessary for the existence of non-principal ultrafilters, in the sense that there is a model of ZF set theory for which all ultrafilters are principal. Thus, one cannot ever explicitly construct or see an example of a non-principal ultrafilter.

As we will see in Theorem 4.2 below, the existence of these non-principal ultrafilters on $\mathbb{N}$ gives rise to "non-principal" pure states on $\mathbb{D}(\ell_2)$ which have no counterpart in finite dimensions. This is what makes the infinite-dimensional Kadison-Singer problem harder than the finite-dimensional problem considered in Section 2: we have to worry about extendability of these "non-principal" pure states that we can never explicitly construct.

Claim 3.6. A filter $\mathcal{F}$ of X is an ultrafilter if and only if it is a *maximal* filter (with respect to inclusion as a collection of sets).

Proof. Suppose $\mathcal{F}$ is an ultrafilter and consider any filter $\mathcal{F}' \supseteq \mathcal{F}$. For any $A \notin \mathcal{F}$, we have $A^c \in \mathcal{F}$ and hence $A^c \in \mathcal{F}'$. But $\mathcal{F}'$ cannot contain both A and A^c, since it is closed under intersections and it omits $\emptyset$. Thus $\mathcal{F}'$ cannot contain any set omitted from $\mathcal{F}$, so $\mathcal{F}$ is maximal.

Let $\mathcal{F}$ be a filter that is not an ultrafilter. Then there is $A \subseteq X$ such that $\mathcal{F}$ contains neither A nor A^c. We will show that $\mathcal{F}$ is not maximal.

Consider adding A to $\mathcal{F}$. To maintain the filter properties, we must also add every set $A \cap Y$ for $Y \in \mathcal{F}$, and every superset of those new sets. If each $A \cap Y \neq \emptyset$ then we claim that the resulting family is a filter. To see this, consider two newly added sets U and U' where $U \supseteq A \cap Y$ and $U' \supseteq A \cap Y'$. Then $U \cap U' \supseteq A \cap (Y \cap Y')$, so $U \cap U'$ was also added.

So $\mathcal{F}$ can be extended to a larger filter (by adding either A or A^c) unless $A \cap Y = \emptyset$ and $A^c \cap Y' = \emptyset$ for some $Y, Y' \in \mathcal{F}$. But this cannot happen: Y and Y' cannot both belong to $\mathcal{F}$ because $Y \cap Y' = \emptyset$. ■

References: Hindman-Strauss [HS98, Theorem 3.6], Tricki.

Recall the finite intersection property, defined in Definition A.7. Obviously every filter has the finite intersection property. The following claim shows that every filter is contained in an ultrafilter.

Claim 3.7. Let $\mathcal{A} \subseteq 2^X$ have the finite intersection property. Then there is an ultrafilter $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{U}$.

Proof. The proof relies on the axiom of choice (via Zorn's lemma). Define

$$\Gamma = \left\{ \mathcal{B} \subseteq 2^X \, : \, \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{B} \text{ has the finite intersection property} \right\}.$$

Obviously Γ is non-empty because $\mathcal{A} \in \Gamma$. Let $\mathcal{C} \subseteq \Gamma$ be a chain. Define $\mathcal{V} \subseteq 2^X$ by $\mathcal{V} = \bigcup \mathcal{C}$.

We claim that $\mathcal{V} \in \Gamma$. Obviously $\mathcal{A} \subseteq \mathcal{V}$ so it remains to show that $\mathcal{V}$ has the finite intersection property. Consider any finite subcollection of $\mathcal{V}$, say $F_1, ..., F_k \in \mathcal{V}$. Because $\mathcal{C}$ is a chain, there exists a $\mathcal{B} \in \mathcal{C}$ such that each $F_i \in \mathcal{B}$. Since $\mathcal{B}$ has the finite intersection property, $\bigcap_{i=1}^k F_i \neq \emptyset$. Thus $\mathcal{V}$ also has the finite intersection property, so $\mathcal{V} \in \Gamma$.

By Zorn's lemma, we may pick a maximal element $\mathcal{U}$ in Γ. So $\mathcal{U}$ is maximal with respect to the finite intersection property. It is easy to see that $\mathcal{U}$ is a filter: if $A \in \mathcal{U}$ and $B \supseteq A$ then $B \in \mathcal{U}$ because $\mathcal{U} \cup \{B\}$ also has the finite intersection property and $\mathcal{U}$ is maximal. So $\mathcal{U}$ is also maximal with respect to being a filter. By Claim 3.6, $\mathcal{U}$ is an ultrafilter. ■

References: Hindman [Hin96, Theorem 2.4], Hindman-Strauss [HS98, Theorem 3.8].

We can now show that non-principal ultrafilters exist. Let $\mathcal{U}$ be the cofinite filter and let $\mathcal{U}' \supseteq \mathcal{U}$ be a filter. $\mathcal{U}'$ cannot contain any finite set F because it also contains F^c, so it would have to contain their empty intersection. So $\mathcal{U}'$ cannot be a principal ultrafilter. Claim 3.7 shows that there exists an ultrafilter $\mathcal{U}' \supseteq \mathcal{U}$, so $\mathcal{U}'$ must be a non-principal ultrafilter.

Claim 3.8. If $\mathcal{U}$ is an ultrafilter that contains any finite set, then $\mathcal{U}$ is a principal ultrafilter.

Proof. Let $S = \{s_1, ..., s_k\}$ be a finite set in $\mathcal{U}$. We claim that $\mathcal{U}$ contains exactly one set $\{s_i\}$. Obviously $\mathcal{U}$ cannot contain distinct sets $\{s_i\}$ and $\{s_j\}$, because their intersection is empty. If $\mathcal{U}$ does not contain any $\{s_i\}$ then $S^c = \{s_1\}^c \cap \cdots \cap \{s_k\}^c \in \mathcal{U}$. That is a contradiction since $S \in \mathcal{U}$. So $\mathcal{U}$ contains some principal ultrafilter $\mathcal{U}_{s_i}$. But $\mathcal{U}_{s_i}$ is a maximal filter, so we must have $\mathcal{U} = \mathcal{U}_{s_i}$. ■

References: [HS98, Theorem 3.7].

Claim 3.9. Let $\mathcal{U}$ be an ultrafilter on X. Suppose that $S = \cup_{i=1}^k S_i \in \mathcal{U}$. Then some $S_i \in \mathcal{U}$.

Proof. Without loss of generality we may assume the S_i are disjoint, because filters are upward-closed. Without loss of generality we may assume that $S = X$, otherwise we may restrict to the induced ultrafilter on S. If $k = 1$ the claim is trivial, so assume $k \geq 2$. If $S_k \in \mathcal{U}$ the claim is true; otherwise $S_k^c = \cup_{i=1}^{k-1} S_i \in \mathcal{U}$. By induction, $\mathcal{U}$ contains some set S_i, $i \in \{1, ..., k-1\}$. ■

3.2 Ultrafilters on $\mathbb{N}$

The set of all ultrafilters on $\mathbb{N}$ is denoted[4] $\beta\mathbb{N} = \{ \mathcal{U} \, : \, \mathcal{U} \text{ is an ultrafilter on } \mathbb{N} \}$. In this section we will define a topology on $\beta\mathbb{N}$. First define:

$$\begin{aligned} \widehat{A} &= \{ \mathcal{U} \in \beta\mathbb{N} \, : \, A \in \mathcal{U} \} \qquad \forall A \subseteq \mathbb{N} \\ \mathcal{A} &= \left\{ \widehat{A} \, : \, A \subseteq \mathbb{N} \right\}. \end{aligned}$$

[4] The notation $\beta\mathbb{N}$ may seem odd, but it originates from the fact that the set of ultrafilters $\beta\mathbb{N}$ is actually the Stone-Čech compactification of $\mathbb{N}$. This document does not directly use this fact.

Claim 3.10. For every $A, B \subseteq \mathbb{N}$, $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$.

Proof. The proof is just a matter of understanding the definitions:

$$\mathcal{U} \in \widehat{A} \cap \widehat{B} \iff A, B \in \mathcal{U} \iff A \cap B \in \mathcal{U} \iff \mathcal{U} \in \widehat{A \cap B}.$$

■

References: Hindman [Hin96, Lemma 2.7(a)], Hindman-Strauss [HS98, Lemma 3.17(a)].

Claim 3.11. For every $A \subseteq \mathbb{N}$, $(\beta\mathbb{N}) \setminus \widehat{A} = \widehat{(A^c)}$. Consequently, $\mathcal{A}$ is closed under complements; that is, the family of complements of members of $\mathcal{A}$ is $\mathcal{A}$ itself.

Proof. We omit the proof, which again follows directly from the definitions. ■

References: Hindman [Hin96, Lemma 2.7(c)], Hindman-Strauss [HS98, Lemma 3.17(c)].

Claim 3.10 shows that $\mathcal{A}$ is closed under finite intersections. Obviously $\bigcup_{\widehat{A} \in \mathcal{A}} \widehat{A} = \beta\mathbb{N}$, so by Fact A.3 we may define the topology on $\beta\mathbb{N}$ to be all unions of members of $\mathcal{A}$. The family $\mathcal{A}$ is a base for the open sets of this topology. By Claim 3.11, the family of complements of members of $\mathcal{A}$ is $\mathcal{A}$ itself so, by Fact A.5, $\mathcal{A}$ is also a base for the closed sets. This implies that every set $\widehat{A}$ is clopen.

Claim 3.12. $\beta\mathbb{N}$ is compact.

Proof. Recall that $\mathcal{A}$ is a base for the closed sets. By Claim A.8, it suffices to show that every subcollection $\mathcal{B} \subseteq \mathcal{A}$ with the finite intersection property has non-empty intersection. Let $\mathcal{S} = \left\{ S \subseteq \mathbb{N} \,:\, \widehat{S} \in \mathcal{B} \right\}$.

We claim that $\mathcal{S}$ has the finite intersection property. To see this, for any $S_1, ..., S_k \in \mathcal{S}$, there exists $\mathcal{U} \in \bigcap_{i=1}^{k} \widehat{S}_i$ (since $\mathcal{B}$ has the finite intersection property), so $S_1, ..., S_k \in \mathcal{U}$, which implies $\bigcap_{i=1}^{k} S_i \in \mathcal{U}$, and hence $\bigcap_{i=1}^{k} S_i$ is not empty.

Claim 3.7 implies that there is an ultrafilter $\mathcal{U} \supseteq \mathcal{S}$. This means that $\mathcal{U} \in \widehat{S}$ for all $S \in \mathcal{S}$, i.e., $\bigcap \mathcal{B}$ is non-empty. ■

References: Hindman [Hin96, Lemma 2.11], Hindman-Strauss [HS98, Theorem 3.18(a)].

Claim 3.13. The principal ultrafilters are dense in $\beta\mathbb{N}$.

Proof. Since $\mathcal{A}$ is a base, it suffices to show that, for every $A \subseteq \mathbb{N}$, $\widehat{A}$ contains a principal ultrafilter. For every $i \in A$, obviously $A \in \mathcal{U}_i$, so $\mathcal{U}_i \in \widehat{A}$. ■

References: Hindman [Hin96, Lemma 2.10], Hindman-Strauss [HS98, Theorem 3.18(e)].

Claim 3.14. $\beta\mathbb{N}$ is Hausdorff.

Proof. Let $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ be distinct. Fix any set $A \in \mathcal{U} \setminus \mathcal{V}$; then $A^c \in \mathcal{V} \setminus \mathcal{U}$. Then $\widehat{A}$ and $\widehat{(A^c)}$ are disjoint open sets with $\mathcal{U} \in \widehat{A}$ and $\mathcal{V} \in \widehat{(A^c)}$. ■

References: Hindman [Hin96, Lemma 2.8], Hindman-Strauss [HS98, Theorem 3.18(a)].

3.3 $\beta\mathbb{N}$ and limits of sequences

Let $a = (a_1, a_2, ...) \in \mathbb{C}^{\mathbb{N}}$ be a bounded sequence (i.e., $a \in \ell_\infty$). The sequence a might not have a limit. We will show that an ultrafilter $\mathcal{U}$ can be used to define a "$\mathcal{U}$-limit".

Definition 3.15. Let $a \in \mathbb{C}^{\mathbb{N}}$ and $\mathcal{U} \in \beta\mathbb{N}$. We say that a point $x \in \mathbb{C}$ is a $\mathcal{U}$-limit of a if, for every neighborhood S of x, we have $\{ \, i \,:\, a_i \in S \, \} \in \mathcal{U}$.

Claim 3.16. If $a \in \mathbb{C}^{\mathbb{N}}$ has a $\mathcal{U}$-limit, it is unique.

Proof. Let x and x' be distinct points in $\mathbb{C}$. Let S and S' be disjoint neighborhoods of x and x' respectively. Then the sets $\{ i : a_i \in S \}$ and $\{ i : a_i \in S' \}$ are disjoint, so they cannot both belong to $\mathcal{U}$. So x and x' cannot both be $\mathcal{U}$-limits. ■

References: Hindman [Hin96, Theorem 2.16(a)], Hindman-Strauss [HS98, Theorem 3.48(a)].

So, if $a \in \mathbb{C}^{\mathbb{N}}$ has a $\mathcal{U}$-limit, we may denote it by $\mathcal{U}\text{-}\lim a$.

Claim 3.17. Let $a \in \mathbb{C}^{\mathbb{N}}$ and let $\mathcal{U} \in \beta\mathbb{N}$ be the principal ultrafilter $\mathcal{U}_j$ for some $j \in \mathbb{N}$. Then $\mathcal{U}\text{-}\lim a = a_j$.

Proof. Fix any neighborhood S of a_j and let $A = \{ i : a_i \in S \}$. Obviously $j \in A$ so $A \in \mathcal{U}$. Thus a_j is a $\mathcal{U}$-limit of a. ■

Claim 3.18. If $a \in \ell_\infty$ then $\mathcal{U}\text{-}\lim a$ exists.

Proof. Let $X \subseteq \mathbb{C}$ be a compact set such that each $a_i \in X$. If $\mathcal{U}\text{-}\lim a$ does not exist then, for every $x \in X$, there must exist an open neighborhood S_x with $\{ i : a_i \in S_x \} \notin \mathcal{U}$. Then $X \subseteq \bigcup_{x \in X} S_x$, so there exists a finite $F \subseteq X$ with $X \subseteq \bigcup_{x \in F} S_x$. Then $\mathbb{N} = \bigcup_{x \in F} \{ i : a_i \in S_x \}$. Claim 3.9 shows that some x has $\{ i : a_i \in S_x \} \in \mathcal{U}$, which is a contradiction. ■

References: Hindman [Hin96, Theorem 2.16(b)], Hindman-Strauss [HS98, Theorem 3.48(b)].

Claim 3.19. Let $\mathcal{U}$ be an ultrafilter and let $I \in \mathcal{U}$. Let $a \in \mathbb{C}^{\mathbb{N}}$ and suppose that $\mathcal{U}\text{-}\lim a$ exists. Define $A = \{ a_i : i \in I \}$. Then $\mathcal{U}\text{-}\lim a \in \overline{A}$, the closure of A.

Proof. Suppose not. Then $X := (\overline{A})^c$ is a neighborhood of $\mathcal{U}\text{-}\lim a$, so $\{ i : a_i \in X \} \in \mathcal{U}$. But $\{ i : a_i \in X \}$ is disjoint from I, so their empty intersection must also be in $\mathcal{U}$, which is a contradiction. ■

Claim 3.20. The $\mathcal{U}\text{-}\lim$ operator is linear. That is, for $a, b \in \mathbb{C}^{\mathbb{N}}$ and $c, d \in \mathbb{C}$, we have $c \cdot \mathcal{U}\text{-}\lim a + d \cdot \mathcal{U}\text{-}\lim b = \mathcal{U}\text{-}\lim\,(ca + db)$ whenever the left-hand side exists.

Proof. Define $l_a = \mathcal{U}\text{-}\lim a$ and $l_b = \mathcal{U}\text{-}\lim b$. For any $\epsilon > 0$,

$$\begin{aligned} \{ i : |ca_i - c \cdot l_a| < |c|\epsilon \} &= \{ i : |a_i - l_a| < \epsilon \} \in \mathcal{U} \\ \{ i : |db_i - d \cdot l_b| < |d|\epsilon \} &= \{ i : |b_i - l_b| < \epsilon \} \in \mathcal{U} \end{aligned}$$

Their intersection is also in $\mathcal{U}$, and hence $\{ i : |(ca_i + db_i) - (c \cdot l_a + d \cdot l_b)| < 2\epsilon \} \in \mathcal{U}$. Taking $\epsilon \downarrow 0$, we see that $c \cdot l_a + d \cdot l_b$ is a $\mathcal{U}$-limit of $ca + db$. ■

References: Tricki.

Claim 3.21. Let $a, b \in \mathbb{C}^{\mathbb{N}}$ and let ab be their pointwise product, i.e., $(ab)_i = a_i \cdot b_i$. Then $\big(\mathcal{U}\text{-}\lim a\big) \cdot \big(\mathcal{U}\text{-}\lim b\big) = \mathcal{U}\text{-}\lim\,(ab)$ whenever the left-hand side exists.

Proof. The proof is similar to the previous proof, and the usual argument that limits are multiplicative (e.g., Rudin [Rud76, Theorem 3.3]). ■

Claim 3.22. Let $a \in \mathbb{C}^{\mathbb{N}}$ and let a^* be its pointwise conjugate, i.e., $(a^*)_i = (a_i)^*$. Then $\big(\mathcal{U}\text{-}\lim a\big)^* = \mathcal{U}\text{-}\lim\,(a^*)$ whenever the left-hand side exists.

Proof. The proof is easy and omitted. ■

As in Example B.2, define $C(\beta\mathbb{N})$ to be the Banach space of continuous, complex-valued functions on $\beta\mathbb{N}$ with the supremum norm $\|f\|_u = \sup \{ |f(\mathcal{U})| : \mathcal{U} \in \beta\mathbb{N} \}$. Since $\beta\mathbb{N}$ is compact, all such functions are bounded (by Fact A.10).

Claim 3.23. The Banach spaces ℓ_∞ and $C(\beta\mathbb{N})$ are isometrically isomorphic.

Proof. For any continuous function $f : \beta\mathbb{N} \to \mathbb{C}$, define $a^f \in \mathbb{C}^{\mathbb{N}}$ by $a^f_i = f(\mathcal{U}_i)$. We claim that $a^f \in \ell_\infty$. Since $\beta\mathbb{N}$ is compact and f is continuous, Fact A.10 shows that $f(\beta\mathbb{N})$ is a compact subset of $\mathbb{C}$. Thus $\left\{\, a^f_i \,:\, i \in \mathbb{N} \,\right\} = \{\, f(\mathcal{U}_i) \,:\, i \in \mathbb{N} \,\}$ is bounded.

Conversely, for any $a \in \ell_\infty$, define $f_a : \beta\mathbb{N} \to \mathbb{C}$ by $f_a(\mathcal{U}) = \mathcal{U}\text{-lim}\, a$. We now argue that f_a is continuous. Let S be a neighborhood of $f_a(\mathcal{U})$. Let $S' \subseteq S$ be a closed neighborhood of $f_a(\mathcal{U})$. Then $I := \{\, i \,:\, a_i \in S' \,\} \in \mathcal{U}$. Since $\widehat{I}$ is open in the topology of $\beta\mathbb{N}$, it is a neighborhood of $\mathcal{U}$. We claim that $f_a(\mathcal{V}) \in S'$ for all $\mathcal{V} \in \widehat{I}$. Define $A := \{\, a_i \,:\, i \in I \,\} \subseteq S'$. Since $I \in \mathcal{V}$, we apply Claim 3.19 to obtain $\mathcal{V}\text{-lim}\, a \in \overline{A}$. Since S' is closed, $\overline{A} \subseteq S'$, so $f_a(\mathcal{V}) \in S'$. Thus $f_a(\widehat{I}) \subseteq S$. By Fact A.11, f_a is continuous.

Next we claim that the maps $a \mapsto f_a$ and $f \mapsto a^f$ are mutually inverse. It is easy to see that $a^{f_a} = a$ for every $a \in \ell_\infty$. Next we claim that, we must have $f_{a^f} = f$ for every $f \in C(\beta\mathbb{N})$. To see this, first note that f and f_{a^f} agree on the principal ultrafilters, because $f_{a^f}(\mathcal{U}_i) = \mathcal{U}_i\text{-lim}\, a^f = a^f_i = f(\mathcal{U}_i)$, by Claim 3.17. By Claim 3.13, the principal ultrafilters are a dense subset of $\beta\mathbb{N}$. Thus f_{a^f} and f are continuous functions that agree on a dense set of the domain, so by Fact A.12 they must be equal.

For any $a \in \ell_\infty$ we have

$$\|a\|_\infty \;=\; \sup_{i\in\mathbb{N}} |a_i| \;=\; \sup_{i\in\mathbb{N}} |f_a(\mathcal{U}_i)| \;=\; \sup_{\mathcal{U}\in\beta\mathbb{N}} |f_a(\mathcal{U})| \;=\; \|f_a\|_u \,,$$

where the third equality holds since the principal ultrafilters are dense in $\mathbb{N}$ (Claim 3.13) and f_a is continuous. ■

4 Ultrafilters on $\mathbb{N}$ and functionals on $\mathbb{D}(\ell_2)$

The main goal of this section is to understand the pure states on $\mathbb{D}(\ell_2)$. To do so, we will use the fact, established in Section 3, that $\mathbb{D}(\ell_2)$ and $C(\beta\mathbb{N})$ are isometrically isomorphic. It turns out that the pure states on $\mathbb{D}(\ell_2)$ are closely related to ultrafilters.

Before discussing states on $\mathbb{D}(\ell_2)$, we must first observe that this is a C^*-algebra. One can see this directly, but instead let us recall that Claim 3.12 shows that $\beta\mathbb{N}$ is compact and Hausdorff, so Example C.6 shows that $C(\beta\mathbb{N})$ is a C^*-algebra. Thus, a consequence of Example B.14 and Claim 3.23 is:

Corollary 4.1. The C^*-algebras $\mathbb{D}(\ell_2)$ and $C(\beta\mathbb{N})$ are isometrically isomorphic.

Let $\operatorname{diag} : \mathbb{D}(\ell_2) \to \ell_\infty$ be the canonical isometry, as described in Example B.14. For any $\mathcal{U} \in \beta\mathbb{N}$, define the functional $f_{\mathcal{U}} : \mathbb{D}(\ell_2) \to \mathbb{C}$ by

$$f_{\mathcal{U}}(D) \;=\; \mathcal{U}\text{-lim}\,(\operatorname{diag} D). \tag{4.1}$$

Theorem 4.2. The pure states on $\mathbb{D}(\ell_2)$ are precisely the functionals of the form $f_{\mathcal{U}}$ for $\mathcal{U} \in \beta\mathbb{N}$.

References: Tanbay [Tan91, pp. 707].

Recall the two-dimensional example in Section 2 in which the only pure states were $f_1(d_1, d_2) = d_1$ and $f_2(d_1, d_2) = d_2$. This is consistent with Theorem 4.2 as Claim 3.8 shows that the only ultrafilters on $\{1, 2\}$ are the principal ultrafilters $\mathcal{U}_1 = \{\{1\}, \{1, 2\}\}$ and $\mathcal{U}_2 = \{\{2\}, \{1, 2\}\}$.

As a step towards Theorem 4.2, we start with a smaller theorem which shows that, given any pure state, we can extract an ultrafilter. Recalling Definition B.15, the projections in $\mathbb{D}(\ell_2)$ are $\{ P_A : A \subseteq \mathbb{N} \}$ where

$$\langle P_A e_i, e_i \rangle = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 4.3. Given any pure state $f : \mathbb{D}(\ell_2) \to \mathbb{C}$, define $\mathcal{U} = \{ A \subseteq \mathbb{N} : f(P_A) = 1 \}$. Then $\mathcal{U}$ is an ultrafilter.

In order to prove these theorems, we require the following characterization of pure states on $\mathbb{D}(\ell_2)$.

Claim 4.4. A (non-zero) linear functional on $\mathbb{D}(\ell_2)$ is a pure state if and only if it is multiplicative.

Proof. Directly from Corollary 4.1, Claim 3.14, Claim 3.12 and Fact C.23. ■

Claim 4.5. Let $f : \mathbb{D}(\ell_2) \to \mathbb{C}$ be a pure state and let $P \in \mathbb{D}(\ell_2)$ be a projection. Then $f(P) \in \{0, 1\}$.

Proof. By Claim 4.4, f is multiplicative. So $f(P) = f(P^2) = f(P)^2$, implying $f(P) \in \{0, 1\}$. ■

Proof (of Theorem 4.3).

- $\mathbb{N} \in \mathcal{U}$ because $f(P_{\mathbb{N}}) = f(1) = 1$.
- $\emptyset \notin \mathcal{U}$ because $P_\emptyset = 0$ and $f(0) = 0$ by linearity.
- Suppose $A \in \mathcal{U}$ and $A \subseteq B$. So $f(P_A) = 1$. Note that $P_B = P_A + P_{B \setminus A}$. By Claim 4.5, linearity and positivity of f, $1 \geq f(P_B) = f(P_A) + f(P_{B\setminus A}) \geq f(P_A) = 1$, so $f(P_B) = 1$.
- Let $A \subseteq \mathbb{N}$ be arbitrary. Then $1 = f(I) = f(P_A) + f(P_{A^c})$. By Claim 4.5, exactly one of $f(P_A)$ and $f(P_{A^c})$ is 1, so exactly one of A or A^c is in $\mathcal{U}$.
- Suppose $A \in \mathcal{U}$ and $B \in \mathcal{U}$. Then $A \cup B \in \mathcal{U}$, so $f(P_A) = f(P_B) = f(P_{A\cup B}) = 1$. By linearity of f,

  $$\begin{aligned} 1 &= f(P_A) = f(P_{A\setminus B}) + f(P_{A\cap B}) \\ 1 &= f(P_B) = f(P_{B\setminus A}) + f(P_{A\cap B}) \\ 1 &= f(P_{A\cup B}) = f(P_{A\setminus B}) + f(P_{B\setminus A}) + f(P_{A\cap B}) \end{aligned}$$

  This is only possible if $f(P_{A\cap B}) = 1$, because $f(P_{A\setminus B}), f(P_{B\setminus A}), f(P_{A\cap B}) \in \{0, 1\}$ by Claim 4.5. So $A \cap B \in \mathcal{U}$.

■

Claim 4.6. For any $\mathcal{U} \in \beta\mathbb{N}$, $f_{\mathcal{U}}$ is a *-homomorphism (cf. Definition C.11).

Proof. Fix $\mathcal{U}$ and define $g : \ell_\infty \to \mathbb{C}$ by $a \mapsto \mathcal{U}\text{-}\lim a$. The function g

- *Is homogeneous*: Claim 3.20 implies that $g(ca) = c \cdot g(a)$ for $c \in \mathbb{C}$ and $a \in \ell_\infty$.
- *Is additive*: Claim 3.20 implies that $g(a + b) = g(a) + g(b)$ for all $a, b \in \ell_\infty$.
- *Is multiplicative*: Claim 3.21 implies that $g(ab) = g(a) \cdot g(b)$ for all $a, b \in \ell_\infty$.
- *Is unital*: Claim 3.19 implies that $g(1) = 1$.
- *Commutes with conjugation*: Claim 3.22 implies that $g(a^*) = g(a)^*$ for all $a \in \ell_\infty$.

$f_{\mathcal{U}}$ has the same properties because $f_{\mathcal{U}}(D) = g(\operatorname{diag} D)$. ■

Proof (of Theorem 4.2). The fact that $f_{\mathcal{U}}$ is a pure state on $\mathbb{D}(\ell_2)$ follows from Claim 4.4 and

Claim 4.6. Next, given any pure state on $\mathbb{D}(\ell_2)$, define $\mathcal{U}$ as in Theorem 4.3. We claim that $f = f_{\mathcal{U}}$, i.e., for every $D \in \mathbb{D}(\ell_2)$, we have $f(D) = \mathcal{U}\text{-lim diag}\, D$. It suffices to consider the case that D is a projection, by Fact B.16 and the fact that f is linear. Thus, assume $D = P_A$ and $d = \text{diag}\, D \in \{0,1\}^{\mathbb{N}}$ has $d_i = 1$ iff $i \in A$. If $f(P_A) = 1$ then $A \in \mathcal{U}$, so $\mathcal{U}\text{-lim}\, d = 1$ by Claim 3.19. Conversely, if $f(P_A) = 0$ then $A^c \in \mathcal{U}$, so $\mathcal{U}\text{-lim}\, d = 0$ by Claim 3.19. ■

4.1 Statement of the Kadison-Singer Problem

Let $f : \mathbb{D}(\ell_2) \to \mathbb{C}$ be a linear functional. A linear functional $g : B(\ell_2) \to \mathbb{C}$ is called an ***extension*** of f to $B(\ell_2)$ if $g(D) = f(D)$ for all $D \in \mathbb{D}(\ell_2)$. Additionally, if f and g are both states then then g is called a ***state extension*** of f. Similarly, if f and g are both pure states then g is called a ***pure state extension*** of f.

The Kadison-Singer problem is:

Problem 1.1. Is it true that, for every pure state $f : \mathbb{D}(\ell_2) \to \mathbb{C}$ on $\mathbb{D}(\ell_2)$, there is a unique functional $g : B(\ell_2) \to \mathbb{C}$ that is a state extension of f?

Claim 4.8 clarifies the problem somewhat. First we require a definition

Definition 4.7. For $H \in B(\ell_2)$, let $E(H) \in \mathbb{D}(\ell_2)$ be the "diagonal of H", i.e., $\langle\, E(H)e_i,\, e_j \,\rangle$ equals $\langle\, He_i,\, e_i \,\rangle$ if $i = j$ and equals zero otherwise. This is also called the ***conditional expectation*** of H. Obviously $E : B(\ell_2) \to \mathbb{D}(\ell_2)$ is linear.

Claim 4.8.

- **(1):** Every state on $\mathbb{D}(\ell_2)$ has a state extension to $B(\ell_2)$. More precisely, if $f : \mathbb{D}(\ell_2) \to \mathbb{C}$ be a state, define $g : B(\ell_2) \to \mathbb{C}$ by $g(H) = f(E(H))$. Then g is a state.
- **(2):** Every pure state on $\mathbb{D}(\ell_2)$ has a pure state extension to $B(\ell_2)$.
- **(3):** Let f be a pure state on $\mathbb{D}(\ell_2)$. If g is the unique pure state extension of f to $B(\ell_2)$, then g is also the unique state extension of f to $B(\ell_2)$.

References: Dixmier [Dix77, Lemma 2.10.1]. For (1), see also Kadison-Ringrose [KR83, Theorem 4.3.13(ii)], Fact C.20, Lemma 5.6. For (2) and (3), see Kadison-Ringrose [KR83, Theorem 4.3.13(iv)].

Proof. We prove only (1). Clearly $g(1) = f(E(1)) = f(1) = 1$. Clearly g is linear because it is the composition of linear functions. It remains to check positivity. If $H \in B(\ell_2)$ is positive then $\langle\, He_i,\, e_i \,\rangle \geq 0$ for all i, as in Example C.10. Thus $E(H)$ is also positive. Since f is positive, we have $g(H) = f(E(H)) \geq 0$. ■

Statements (2) and (3) together show that Problem 1.1 is equivalent after replacing the words "state extension" with "pure state extension".

5 Reduction to the paving problem

In this section we show that the Kadison-Singer conjecture is equivalent to Anderson's paving conjecture. The argument is copied nearly verbatim from Paulsen and Raghupathi [PR08].

5.1 Generic results about C^*-algebras

Fix real numbers a, b with $0 < a < 1 < b$. Let $\mathcal{B}$ be a unital C^*-algebra. Define

$$\mathcal{P}[a,b] \;=\; \{\, P \in \mathcal{B} \,:\, aI \preceq P \preceq bI \,\}.$$

Note that every element of $\mathcal{P}[a,b]$ is self-adjoint and positive. The set $\mathcal{P}[a,b]$ is closed and convex.

Claim 5.1 ([PR08, Theorem 2.1]). Let $\mathcal{B}$ be a unital C^*-algebra and let $s_i : \mathcal{B} \to \mathbb{C}$ (for $i = 1,2$) be states. The following are equivalent:

1: $s_1 = s_2$,

2: $s_1(p)s_2(p^{-1}) \geq 1$ for every positive, invertible $p \in \mathcal{B}$,

3: $s_1(p)s_2(p^{-1}) \geq 1$ for every $p \in \mathcal{P}[a,b]$.

Proof. (1) $\Rightarrow$ (2): Suppose that $s : \mathcal{B} \to \mathbb{C}$ is a state and that $q \in \mathcal{B}$ is positive and invertible. For any $t \in \mathbb{R}$, the operator $tq + q^{-1}$ is self-adjoint, so $(tq + q^{-1})^2$ is positive, so

$$0 \;\leq\; s\big((tq + q^{-1})^2\big) \;=\; t^2 s(q^2) + 2t + s(q^{-2}).$$

Since this quadratic function of t is non-negative, its discriminant $4 - 4s(q^2)s(q^{-2})$ must be non-positive. Thus $s(q^2)s(q^{-2}) \geq 1$. This proves (2) by substituting any $p \in \mathcal{P}[a,b]$ for $q^2 = p$ and setting $s = s_1 = s_2$.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $h \in \mathcal{B}$ be self-adjoint. Define the function $f(t) = s_1(e^{th})s_2(e^{-th})$. For real t in a sufficiently small neighborhood of 0, we have $e^{th} \in \mathcal{P}[a,b]$, so (3) implies $f(t) \geq 1$ in that neighborhood. But $f(0) = 1$, so 0 is a minimizer and $f'(0) = 0$. One can show that

$$\frac{df}{dt} \;=\; s_1\big(he^{th}\big)s_2(e^{-th}) + s_2\big(-he^{-th}\big)s_1(e^{th})$$

Thus $0 = f'(0) = s_1(h) - s_2(h)$. That is, $s_1(h) = s_2(h)$ for all self-adjoint h, so $s_1 = s_2$ by Claim C.13. ■

Let $\mathcal{S} \subseteq \mathcal{B}$ be an operator system (see Definition C.12). For any state $s : \mathcal{S} \to \mathbb{C}$ and any self-adjoint $p \in \mathcal{B}$, define

$$\begin{aligned} \ell_s(p) &\;=\; \sup\{\, s(q) \,:\, p \succeq q \in \mathcal{S} \,\} \\ u_s(p) &\;=\; \inf\{\, s(q) \,:\, p \preceq q \in \mathcal{S} \,\} \end{aligned}$$

Here q is implicitly assumed to be self-adjoint, so $s(q) \in \mathbb{R}$. Clearly $\ell_s(p) \leq u_s(p)$, since for every $q \preceq p \preceq q'$, we have $f(q) \leq f(q')$ by Claim C.15.

Claim 5.2. Let $s' : \mathcal{B} \to \mathbb{C}$ be any state extension of the state $s : \mathcal{S} \to \mathbb{C}$. Then, for every self-adjoint $p \in \mathcal{B}$, we have

$$\ell_s(p) \;\leq\; s'(p) \;\leq\; u_s(p).$$

Proof. Since s' is a state, for any $q \in \mathcal{S}$ with $q \preceq p$ we have $s'(q) \leq s'(p)$ by Claim C.15. Since s' extends s we have $s'(q) = s(q)$. Thus $s(q) \leq s'(p)$. Taking the supremum over q proves the first inequality. The other inequality is similar. ■

Claim 5.3. For any $p \in \mathcal{S}$ we have $\ell_s(p) = u_s(p) = s(p)$.

Proof. Taking $q = p$ in the definition of ℓ_s and u_s, we see that $\ell_s(p) \geq s(p)$ and $u_s(p) \leq s(p)$. The reverse direction $\ell_s(p) \leq u_s(p)$ was noted above. ■

Claim 5.4. For any $p \in \mathcal{B}$ and $q \in \mathcal{S}$ we have

$$\ell_s(p+q) = \ell_s(p) + \ell_s(q) \qquad \text{and} \qquad u_s(p+q) = u_s(p) + u_s(q).$$

Claim 5.5. $\ell_s(p) = -u_s(-p)$.

Next we require Krein's extension theorem.

Lemma 5.6. Let $s : \mathcal{S} \to \mathbb{C}$ be a state and let $p \in \mathcal{B}$ be self-adjoint. For every t with $\ell_s(p) \leq t \leq u_s(p)$ there exists a state $s_t : \mathcal{B} \to \mathbb{C}$ that extends s and satisfies $s_t(p) = t$.

References: Kadison-Ringrose [KR83, Theorem 4.3.13(ii)], Krein [Kre40], Paulsen-Raghupathi [PR08, Proposition 2.2].

Proof. If $p \in \mathcal{S}$ then $\ell_s(p) = u_s(p)$ so there is nothing to prove, so assume $p \notin \mathcal{S}$. Define the operator system $\mathcal{T} = \{\, a + \lambda p \,:\, a \in \mathcal{S}, \lambda \in \mathbb{C} \,\}$. Fix t satisfying the condition of the claim, and define the linear functional $f : \mathcal{T} \to \mathbb{C}$ by $f(a + \lambda p) = s(a) + \lambda t$.

Consider any positive operator $a + \lambda p \in \mathcal{T}$. Positivity implies that it is self-adjoint, which in turn implies that a is self-adjoint (since p is). Furthermore, λ must be real.

We claim that $f(a + \lambda p) \geq 0$, i.e., f is a positive linear functional on $\mathcal{T}$.

Case 1: $\lambda > 0$. Then $p \succeq -a/\lambda$. Thus, by definition of $\ell_s(p)$, we have $s(-a/\lambda) \leq \ell_s(p) \leq t$, where the second inequality is by our hypothesis on t. Thus $f(a+\lambda p) = s(a)+\lambda t \geq s(a)+\lambda s(-a/\lambda) = 0$.

Case 2: $\lambda < 0$. The argument is similar.

This proves the claim. Furthermore, $1 \in \mathcal{S}$ so $f(1) = s(1) = 1$, and so f is a state on $\mathcal{T}$. By Fact C.20, f can be extended to a state on $\mathcal{B}$. ■

Theorem 5.7 ([PR08, Theorem 2.3]). Let $\mathcal{B}$ be a unital C^*-algebra, let $\mathcal{S} \subseteq \mathcal{B}$ be an operator system and let $s : \mathcal{S} \to \mathbb{C}$ be a state. The following are equivalent.

1: s extends uniquely to a state on $\mathcal{B}$,

2: for every self-adjoint $p \in \mathcal{B}$, $\ell_s(p) = u_s(p)$,

3: for every positive, invertible $p \in \mathcal{B}$, $\ell_s(p)\ell_s(p^{-1}) \geq 1$,

4: for every $p \in \mathcal{P}[a,b]$, $\ell_s(p)\ell_s(p^{-1}) \geq 1$.

Proof. (1) $\Leftrightarrow$ (2): This follows directly from Claim 5.2 and Lemma 5.6. See also [KR83, Theorem 4.3.13(iii)].

(1) and (2) $\Rightarrow$ (3): Let s_1 be the unique state extension of s. Then, for every $p \in \mathcal{B}$, $s_1(p) = \ell_s(p)$. So $\ell_s(p)\ell_s(p^{-1}) = s_1(p)s_1(p^{-1}) \geq 1$, by Claim 5.1.

(3) $\Rightarrow$ (4): This is trivial.

(4) $\Rightarrow$ (1): Suppose s_1 and s_2 are two state extensions of s. Claim 5.2 yields $s_1(p)s_2(p^{-1}) \geq \ell_s(p)\ell_s(p^{-1}) \geq 1$ for any $p \in \mathcal{P}[a,b]$, by condition (4). By Claim 5.1, $s_1 = s_2$. ■

5.2 The Kadison-Singer problem

First we need a simple fact about Hilbert spaces.

Lemma 5.8. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{K})$ be self-adjoint with A positive and invertible. Then there exists $\delta > 0$ such that $H + \delta P_{\mathcal{K}} \succeq 0$, where $P_{\mathcal{K}}$ denotes the orthogonal projection onto $\mathcal{K}$.

Proof. Let $X = A^{-1/2}B$. Then, by Cauchy-Schwarz,

$$\begin{aligned}
&\left\langle \begin{pmatrix} A & B \\ B^* & C + \delta I_{\mathcal{K}} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \right\rangle \\
&\quad = \langle Ah, h \rangle + \langle A^{1/2}Xk, h \rangle + \langle X^*A^{1/2}h, k \rangle + \langle Ck, k \rangle + \delta \|k\|^2 \\
&\quad \geq \left\|A^{1/2}h\right\|^2 - 2\|Xk\| \left\|A^{1/2}h\right\| - \|C\| \|k\|^2 + \delta \|k\|^2 \\
&\quad \geq \left(\left\|A^{1/2}h\right\| - \|Xk\| \right)^2 + (\delta - \|C\| - \|X\|^2) \|k\|^2 .
\end{aligned}$$

This is non-negative if $\delta \geq \|C\| + \|X\|^2$. ■

This next lemma contains the heart of the proof: the connection between unique extensions and matrix pavings.

Lemma 5.9. Let $\mathcal{U} \in \beta\mathbb{N}$ and let $f_{\mathcal{U}} : \mathbb{D}(\ell_2) \to \mathbb{C}$ be the corresponding pure state (defined in (4.1)). Fix any self-adjoint $H \in B(\ell_2)$. For $t \in \mathbb{R}$, the following two conditions are equivalent:

1: $\ell_{f_{\mathcal{U}}}(H) = u_{f_{\mathcal{U}}}(H) = t$,

2: $\forall \epsilon > 0$, $\exists A \in \mathcal{U}$ such that $(t-\epsilon)P_A \preceq P_A H P_A \preceq (t+\epsilon)P_A$.

Proof. (2) $\Rightarrow$ (1): Fix ϵ and A satisfying (2). Let $f : B(\ell_2) \to \mathbb{C}$ be any state extension of $f_{\mathcal{U}}$. Then $f(P_A) = f_{\mathcal{U}}(P_A) = 1$ and $f(P_{A^c}) = f_{\mathcal{U}}(P_{A^c}) = 0$, as argued in the proof of Theorem 4.2. Then $f(P_{A^c}H) = 0$ by Corollary C.19. Since $I = P_A + P_{A^c}$, linearity implies $f(H) = f(P_AH) + f(P_{A^c}H) = f(P_AH)$. Repeating this argument, $f(P_AHP_A) = f(H)$. Thus, by (2) and Claim C.15, we have $t-\epsilon \leq f(H) \leq t+\epsilon$. Taking $\epsilon \downarrow 0$, we have shown that $f(H) = t$ for every state extension f of $f_{\mathcal{U}}$. By Lemma 5.6, we must have $\ell_{f_{\mathcal{U}}}(H) = u_{f_{\mathcal{U}}}(H) = t$.

(1) $\Rightarrow$ (2): Fix any $\epsilon > 0$. There exist self-adjoint $D_1, D_2 \in \mathbb{D}(\ell_2)$ with $D_1 \preceq H \preceq D_2$ and $t - \epsilon/2 \leq f_{\mathcal{U}}(D_1) \leq f_{\mathcal{U}}(D_2) \leq t + \epsilon/2$. (Since D_i is self-adjoint, $f_{\mathcal{U}}(D_i)$ is real, by Claim C.17.)

Recall Definition 3.15. For $j \in \{1,2\}$, since $f_{\mathcal{U}}(D_j) = \mathcal{U}\text{-lim}\,(\operatorname{diag} D_j)$ we have $A_j := \{\, i : |(\operatorname{diag} D_j)_i - f_{\mathcal{U}}(D_j)| < \epsilon/2 \,\} \in \mathcal{U}$. Then $A := A_1 \cap A_2 \in \mathcal{U}$. So, for all $i \in A$,

$$(\operatorname{diag} D_1)_i > t - \epsilon \quad \text{and} \quad (\operatorname{diag} D_2)_i < t + \epsilon.$$

Thus

$$P_A\big(D_1 - (t-\epsilon)I\big)P_A \succ 0 \quad \text{and} \quad P_A\big((t+\epsilon)I - D_2\big)P_A \succ 0.$$

Apply Lemma 5.8 to $D_1 - (t-\epsilon)I$ where $\mathcal{H}$ is the subspace corresponding to A and $\mathcal{K} = \mathcal{H}^\perp$. (We only apply the lemma to this diagonal operator.) Then there exists $\delta_1 > 0$ such that $D_1 - (t-\epsilon)I + \delta_1 P_{A^c} \succeq 0$. Rearranging and multiplying by P_A on both sides, $P_AD_1P_A \succeq (t-\epsilon)P_A$. A similar argument shows $P_AHP_A \preceq (t+\epsilon)P_A$. ■

Recall from Definition 4.7 that $E(H)$ is the "diagonal" of H.

Theorem 5.10. Let $\mathcal{U} \in \beta\mathbb{N}$ and let $f_{\mathcal{U}} : \mathbb{D}(\ell_2) \to \mathbb{C}$ be the corresponding pure state. The following are equivalent:

1: $f_{\mathcal{U}}$ extends uniquely to a state on $B(\ell_2)$.

2: for every self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$, we have $\ell_{f_{\mathcal{U}}}(H) = 0$.

3: for every self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$ and every $\epsilon > 0$, there exists $A \in \mathcal{U}$ with $-\epsilon P_A \preceq P_A H P_A \preceq \epsilon P_A$.

Proof. (1) ⇒ (2). Consider any self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$. Applying Theorem 5.7 with $\mathcal{S} = \mathbb{D}(\ell_2)$ and $s = f_\mathcal{U}$, we have $\ell_{f_\mathcal{U}}(H) = u_{f_\mathcal{U}}(H)$. By Claim 4.8 (1), there exists a state $g : B(\ell_2) \to \mathbb{C}$ such that $g(H) = f(E(H)) = f(0)$, which is 0 by linearity of f. By Claim 5.2, we must have $\ell_{f_\mathcal{U}}(H) = u_{f_\mathcal{U}}(H) = 0$.

(2) ⇒ (1). Fix a self-adjoint $A \in B(\ell_2)$. By (2), $\ell_{f_\mathcal{U}}(A - E(A)) = 0$. By Claim 5.4 we have

$$\ell_{f_\mathcal{U}}(A) \;=\; \ell_{f_\mathcal{U}}(A - E(A)) + \ell_{f_\mathcal{U}}(E(A)) \;=\; \ell_{f_\mathcal{U}}(E(A)).$$

Similarly, (2) implies $\ell_{f_\mathcal{U}}(E(A) - A) = 0$, so

$$\ell_{f_\mathcal{U}}(-A) \;=\; \ell_{f_\mathcal{U}}(E(A) - A) + \ell_{f_\mathcal{U}}(-E(A)) \;=\; \ell_{f_\mathcal{U}}(-E(A)).$$

By Claim 5.5, this implies $u_{f_\mathcal{U}}(A) = u_{f_\mathcal{U}}(E(A))$. Claim 5.3 shows that $u_{f_\mathcal{U}}(E(A)) = \ell_{f_\mathcal{U}}(E(A))$. Combining these equalities we obtain $\ell_{f_\mathcal{U}}(A) = u_{f_\mathcal{U}}(A)$. Since this holds for all self-adjoint $A \in B(\ell_2)$, Theorem 5.7 implies that (1) holds.

(2) ⇒ (3): Let $H \in B(\ell_2)$ be self-adjoint and satisfy $E(H) = 0$. By (2) we have $\ell_{f_\mathcal{U}}(H) = u_{f_\mathcal{U}}(H) = 0$. Lemma 5.9 with $t = 0$ directly implies (3).

(3) ⇒ (1): Fix any self-adjoint $S \in B(\ell_2)$ and any $\epsilon > 0$. Let $H = S - E(S)$. By (3), there exists $A \in \mathcal{U}$ with $-(\epsilon/2)P_A \preceq P_A H P_A \preceq (\epsilon/2)P_A$. Let $d := \operatorname{diag} E(S) = \operatorname{diag} S$ and $t := f_\mathcal{U}(E(S))$. By the definition of $\mathcal{U}$-limits and the definition $f_\mathcal{U}(E(S)) = \mathcal{U}\text{-lim}\, d$, we have $B := \{\, i \in \mathbb{N} : |d_i - f_\mathcal{U}(E(S))| < \epsilon/2 \,\} \in \mathcal{U}$. Letting $C := A \cap B \in \mathcal{U}$, we have

$$\begin{aligned} -(\epsilon/2)P_C \;&\preceq\; P_C H P_C \;\preceq\; (\epsilon/2)P_C \\ -(\epsilon/2)P_C \;&\preceq\; P_C E(S) P_C - tP_C \;\preceq\; (\epsilon/2)P_C. \end{aligned}$$

Adding these inequalities, using $H = S - E(S)$, then rearranging, we obtain

$$(t - \epsilon)P_C \;\preceq\; P_C S P_C \;\preceq\; (t + \epsilon)P_C.$$

Since such a C exists for all $\epsilon > 0$, Lemma 5.9 implies $\ell_{f_\mathcal{U}}(H) = u_{f_\mathcal{U}}(H) = t$. Since this holds for all self-adjoint $H \in B(\ell_2)$, Theorem 5.7 implies (1). ■

Henceforth let $[r]$ denote $\{1, ..., r\}$.

Theorem 5.11. The following are equivalent:

1: **[The Kadison-Singer Problem (Problem 1.1)]**
every pure state $f_\mathcal{U} : \mathbb{D}(\ell_2) \to \mathbb{C}$ extends uniquely to a state on $B(\ell_2)$.

2: for every $\epsilon > 0$ and for every self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$, there exists $r \in \mathbb{N}$ and a partition $\{A_1, ..., A_r\}$ of $\mathbb{N}$ such that $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for all $i \in [r]$.

3: **[Anderson's Infinite-Dimensional Paving Conjecture]**
for every $\epsilon > 0$ there exists $r \in \mathbb{N}$ such that for every self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $\mathbb{N}$ such that $\|P_{A_i} H P_{A_i}\| \le \epsilon \|H\|$ for all $i \in [r]$.

Proof. (1) ⇒ (2): Fix a self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$ and $\epsilon > 0$. For every $\mathcal{U} \in \beta\mathbb{N}$, since (1) holds, Theorem 5.10 implies that there exists $A_\mathcal{U} \in \mathcal{U}$ with $-\epsilon P_{A_\mathcal{U}} \preceq P_{A_\mathcal{U}} H P_{A_\mathcal{U}} \preceq \epsilon P_{A_\mathcal{U}}$. Then $\left\{\, \widehat{A_\mathcal{U}} : \mathcal{U} \in \beta\mathbb{N} \,\right\}$ is a collection of open sets that cover $\beta\mathbb{N}$. By Claim 3.12, there is a finite subcollection $\left\{\widehat{A_1}, ..., \widehat{A_k}\right\}$ that covers $\beta\mathbb{N}$ and satisfies $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for all i.

To establish (2), we first note that $\bigcup_{i=1}^k A_i = \mathbb{N}$. Indeed, for any $j \in (\bigcup_{i=1}^k A_i)^c$, the principal ultrafilter $\mathcal{U}_j$ cannot contain any A_i, so $\mathcal{U}_j \notin \bigcup_{i=1}^k \widehat{A_i}$, which is a contradiction. So $\{A_1, ..., A_k\}$ is a cover of $\mathbb{N}$, but it might not be a partition because the sets need not be disjoint. Nevertheless, the partition of $\mathbb{N}$ obtained by intersecting those A_i's in all possible ways satisfies (2) with $r \leq 2^k$.

(2) $\Rightarrow$ (1): Fix an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$, a self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$, and $\epsilon > 0$. By (2), there exists a partition $\{A_1, ..., A_r\}$ of $\mathbb{N}$ such that $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for all i. By Claim 3.9, some $A_i \in \mathcal{U}$. Since this holds for all H and all ϵ, Theorem 5.10 implies that $f_{\mathcal{U}}$ extends uniquely to a state on $B(\ell_2)$. Since $\mathcal{U}$ is arbitrary, this implies (1).

(3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): We prove the contrapositive. Suppose (3) does not hold. That is, there exists an $\epsilon > 0$ such that for every $r \in \mathbb{N}$, there exists a self-adjoint $H_r \in B(\ell_2)$ with $E(H_r) = 0$ such that no partition $\{A_1, ..., A_r\}$ of $\mathbb{N}$ satisfies $\|P_{A_i} H_r P_{A_i}\| \leq \epsilon \|H_r\|$ for all i. By rescaling, we may assume that $\|H_r\| = 1$. Let $H = H_1 \oplus H_2 \oplus H_3 \oplus \cdots$. Then H is self-adjoint, $E(H) = 0$ and $\|H\| = 1$, so $H \in B(\ell_2)$. Suppose (2) holds. Then there exists $r \in \mathbb{N}$ and a partition $\{A_1, ..., A_r\}$ satisfying $-\epsilon P_{A_i} \preceq P_{A_i} H P_{A_i} \preceq \epsilon P_{A_i}$ for all i. In particular, restricting to the r^{th} part of the direct sum, $-\epsilon P_{A_i} \preceq P_{A_i} H_r P_{A_i} \preceq \epsilon P_{A_i}$ for all i. Equivalently, $\|P_{A_i} H_r P_{A_i}\| \leq \epsilon \|H_r\|$ for all $i = 1, ..., r$. This is a contradiction, so (2) cannot hold. ■

6 Reduction to finite dimensional paving

Theorem 6.1. The following are equivalent:

1: [**Anderson's Infinite-Dimensional Paving Conjecture**]
for every $\epsilon > 0$ there exists $r \in \mathbb{N}$ such that for every self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $\mathbb{N}$ such that $\|P_{A_i} H P_{A_i}\| \leq \epsilon \|H\|$ for all $i \in [r]$.

2: [**Anderson's Finite-Dimensional Paving Conjecture**]
for every $\epsilon > 0$ there exists $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every self-adjoint operator H on ℓ_2^n with $E(H) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} H P_{A_i}\| \leq \epsilon \|H\|$ for all $i \in [r]$.

References: Tanbay [Tan91, Proposition 1.2], Casazza et al. [CFTW06, Theorem 2.3], Halpern et al. [HKW87, Theorem 2.3].

First we need a technical claim.

Claim 6.2 ([CFTW06, Proposition 2.2]). Suppose that $\mathcal{A}^n = \{A_1^n, ..., A_r^n\}$ is a partition of $[n]$, for each $n \geq 1$. Then there exists a partition $\mathcal{B} = \{B_1, ..., B_r\}$ of $\mathbb{N}$ such that, for every $j \in [r]$ and every $\ell \in \mathbb{N}$, there exists $n_{j,\ell} \in \mathbb{N}$ such that the ℓ smallest elements of B_j are all contained in $A_j^{n_{j,\ell}}$.

Proof. We construct a sequence of integers $1 \leq k_1 \leq k_2 \leq \cdots$ satisfying $k_i \geq i$ and satisfying the following property. For each $m \geq 1$, let $j_m \in [r]$ be the index such that $m \in A_{j_m}^{k_m}$; then $m \in A_{j_m}^{k_p}$ for all $p \geq m$.

Given the sequence of k_i's, define $B_j = \{ m : j_m = j \}$. It is obvious that $\{B_1, ..., B_r\}$ is a partition of $\mathbb{N}$. Now fix $j \in [r]$, let I be the ℓ smallest elements of B_j, and let p be the largest element of I. For any $m \in I$ we have $j_m = j$, so $m \in A_j^{k_m}$ and $m \in A_j^{k_p}$. Taking $n = k_p$, we have $I \subseteq A_j^n$.

We construct the sequence of k_i's as follows. Let $S_0 = \mathbb{N}$. Since S_0 is infinite, there must exist a $j_1 \in [r]$ and an infinite subset $S_1 \subseteq S_0$ such that 1 is contained in every set $A_{j_1}^n$ for all $n \in S_1$. Since S_1 is infinite, there must exist a $j_2 \in [r]$ and an infinite subset $S_2 \subseteq S_1$ such that 2 is contained in every set $A_{j_2}^n$ for all $n \in S_2$. Inductively define the chain $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$.

Now define k_m to be the smallest element of S_m. Clearly $k_m \geq m$. For every $m \geq 1$ we have $m \in A_{j_m}^n$ for all $n \in S_m$; in particular $m \in A_{j_m}^{k_m}$. Furthermore, for each $p \geq m$, $k_p \in S_p \subseteq S_m$, so $m \in A_{j_m}^{k_p}$, as required. ■

Proof (of Theorem 6.1). (1) ⇒ (2): Trivial.

(2) ⇒ (1): Fix $\epsilon > 0$ and let r be the integer whose existence is guaranteed by (2). Fix a self-adjoint $H \in B(\ell_2)$ with $E(H) = 0$. For every $n \geq 1$, let T_n be the $n \times n$ matrix with $(T_n)_{i,j} = \langle Te_i, e_j \rangle$. Clearly $\|T_n\| \leq \|T\|$. By (2), there exists a partition $\mathcal{A}^n = \{A_1, ..., A_1\}$ of $[n]$ such that $\|P_{A_i} T_n P_{A_i}\| \leq (\epsilon/2) \|T_n\|$ for all $i \in [r]$. Given these partitions $\mathcal{A}^1, \mathcal{A}^2, ...$, Claim 6.2 produces a partition $\{B_1, ..., B_r\}$ of $\mathbb{N}$ and integers $n_{j,\ell}$.

Fix $j \in [r]$ and $\ell \in \mathbb{N}$. Let I_ℓ be the ℓ smallest elements of A_j and let p be the largest element of I_ℓ. As $\|P_{I_\ell} T P_{I_\ell}\| \to \left\|P_{A_j} T P_{A_j}\right\|$ monotonically, for sufficiently large ℓ we have

$$\begin{aligned}
\left\|P_{A_j} T P_{A_j}\right\| &\leq 2\left\|P_{I_j} T P_{I_j}\right\| \\
&= 2\left\|P_{I_j} P_{A_j^{n_{j,\ell}}} T P_{A_j^{n_{j,\ell}}} P_{I_j}\right\| \\
&\leq 2\left\|P_{A_j^{n_{j,\ell}}} T P_{A_j^{n_{j,\ell}}}\right\| \\
&\leq 2(\epsilon/2)\|T_n\| \\
&\leq \epsilon\|T\|,
\end{aligned}$$

as required. ■

6.1 Finite-dimensional paving conjectures

Let $\mathbb{C}^{n\times n}$ be the set of complex $n \times n$ matrices. Recall that $E(A)$ is the matrix containing the diagonal entries of A (Definition 4.7), and P_{A_i} is the diagonal projection onto the set A_i (Definition B.15).

Theorem 6.3. The following are equivalent:

1: the Kadison-Singer problem has a positive solution,

2: $\forall \epsilon > 0,\ \exists r \in \mathbb{N},\ \forall n \in \mathbb{N}$ and for every self-adjoint matrix $A \in \mathbb{C}^{n\times n}$ with $E(A) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} A P_{A_i}\| \leq \epsilon \|A\|\ \ \forall i \in [r]$.

3: $\forall \epsilon > 0,\ \exists r \in \mathbb{N},\ \forall n \in \mathbb{N}$ and for every matrix $A \in \mathbb{C}^{n\times n}$ with $E(A) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} A P_{A_i}\| \leq \epsilon \|A\|\ \ \forall i \in [r]$.

4: $\forall \epsilon > 0,\ \exists r \in \mathbb{N},\ \forall n \in \mathbb{N}$ and for every $R \in \mathbb{C}^{n\times n}$ with $R = R^* = R^{-1}$ (i.e., a unitary reflection matrix) and $E(R) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} R P_{A_i}\| \leq \epsilon\ \forall i \in [r]$.

5: $\forall \epsilon > 0,\ \exists r \in \mathbb{N},\ \forall n \in \mathbb{N}$ and for every $Q \in \mathbb{C}^{n\times n}$ with $Q = Q^* = Q^2$ (i.e., a complex orthogonal projection matrix) and $E(Q) = I/2$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} Q P_{A_i}\| \leq (1+\epsilon)/2\ \forall i \in [r]$.

6: $\forall \epsilon > 0$, $\forall$ **even** $N \geq 2$, $\exists r \in \mathbb{N},\ \forall n \in \mathbb{N}$ and for every $Q \in \mathbb{C}^{n\times n}$ with $Q = Q^* = Q^2$ and $Q_{i,i} \in [0, 1/N]$ for all i, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} Q P_{A_i}\| \leq (1+\epsilon)/N\ \forall i \in [r]$.

7: $\forall \epsilon > 0$, $\forall$ **even** $N \geq 2$, $\exists r \in \mathbb{N}$, $\forall d \in \mathbb{N}$ and for all $v_1, ..., v_n \in \mathbb{C}^d$ satisfying $\sum_j v_j v_j^* = I$ and $v_j^* v_j \leq 1/N$ for all $j \in [n]$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that we have $\left\|\sum_{j\in A_i} v_j v_j^*\right\| \leq (1+\epsilon)/N\ \forall i \in [r]$.

References: Portions are from Casazza et al. [CEKP07], Theorems 2 and 3.

Proof. (1) $\Leftrightarrow$ (2): By Theorem 6.1.

(3) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Given $A \in \mathbb{C}^{n\times n}$ we may write $A = M + iN$ where $M = (A + A^*)/2$, $N = i(A^* - A)/2$. Note that M and N are self-adjoint and $\|M\|, \|N\| \leq \|A\|$. If $E(A) = 0$ then $E(M) = E(N) = 0$. Fix any $\epsilon > 0$. By (2), there exist partitions $\{A_1, ..., A_r\}$ and $\{B_1, ..., B_r\}$ of $[n]$ such that $\|P_{A_i} M P_{A_i}\| \leq (\epsilon/2)\|M\|$ and $\|P_{B_i} N P_{B_i}\| \leq (\epsilon/2)\|N\|$. Then $\{C_{i,j} := A_i \cap B_j\}_{i,j}$ is a partition of $[n]$ with

$$\begin{aligned}
\left\|P_{C_{i,j}} A P_{C_{i,j}}\right\| &\leq \left\|P_{C_{i,j}} M P_{C_{i,j}}\right\| + \left\|P_{C_{i,j}} N P_{C_{i,j}}\right\| \\
&\leq \|P_{A_i} M P_{A_i}\| + \left\|P_{B_j} N P_{B_j}\right\| \\
&\leq (\epsilon/2)(\|M\| + \|N\|) \\
&\leq \epsilon \|A\|
\end{aligned}$$

(2) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (2): Let $A \in \mathbb{C}^{n\times n}$ be self-adjoint with $E(A) = 0$. Since (2) is invariant under scaling A, we may assume $\|A\| = 1$. Then $A^2 \preceq I$, so we may define

$$R = \begin{pmatrix} A & \sqrt{I - A^2} \\ \sqrt{I - A^2} & -A \end{pmatrix}.$$

It is easy to check that $R = R^*$ and $R^2 = I$, implying $R = R^* = R^{-1}$. All eigenvalues of R are ± 1, so $\|R\| = 1 = \|A\|$. Furthermore, $E(R) = 0$. By (4), there exists a partition $\{A_1, ..., A_r\}$ of $[2n]$ such that $\|P_{A_i} R P_{A_i}\| \leq \epsilon$ for all i. Then $\{A_1 \cap [n], ..., A_r \cap [n]\}$ is a partition of $[n]$ such that $\|P_{A_i} A P_{A_i}\| \leq \epsilon$ for all i.

(4) $\Rightarrow$ (5): Given Q with $Q = Q^* = Q^2$, let $R = 2Q - I$. Clearly $R = R^*$, and $R^2 = 4Q^2 - 4Q + I = I$, implying $R = R^* = R^{-1}$. Furthermore if $E(Q) = I/2$ then $E(R) = 0$. By (4), there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} R P_{A_i}\| \leq \epsilon$ for all i. Then

$$\|P_{A_i} Q P_{A_i}\| = \|P_{A_i}(I + R) P_{A_i}\|/2 \leq \left\|P_{A_i}^2\right\|/2 + \|P_{A_i} R P_{A_i}\|/2 \leq (1+\epsilon)/2.$$

(5) $\Rightarrow$ (4): Given R with $R = R^* = R^{-1}$ define $Q = (I + R)/2$. Clearly $Q = Q^*$ and $Q^2 = (I + 2R + R^2)/4 = (I + R)/2 = Q$. If additionally $E(R) = 0$ then $E(Q) = I/2$. By (5), there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} Q P_{A_i}\| \leq (1+\epsilon)/2\ \forall i \in [r]$. Since $0 \preceq Q$, we obtain

$$0 \preceq P_{A_i} Q P_{A_i} \preceq (1+\epsilon) P_{A_i}/2.$$

Using $R = 2Q - I$, we get

$$-P_{A_i} \preceq P_{A_i} R P_{A_i} \preceq \epsilon P_{A_i}.$$

This upper-bound is satisfactory but this lower-bound is not.

So apply the same argument to $-R$: define $Q_1 = (I - R)/2$, so $Q_1 = Q_1^* = Q_1^2$ and $E(Q_1) = I/2$. By (5), there exists a partition $\{B_1, ..., B_r\}$ of $[n]$ such that

$$-P_{B_i} \preceq P_{B_i}(-R)P_{B_i} \preceq \epsilon P_{B_i}.$$

Define the partition $\{C_{i,j} := A_i \cap B_j\}_{i,j}$ of $[n]$. We have shown that:

$$\begin{aligned} P_{C_{i,j}} R P_{C_{i,j}} &\preceq P_{A_i} R P_{A_i} \preceq \epsilon P_{A_i} \\ P_{C_{i,j}}(-R) P_{C_{i,j}} &\preceq P_{B_i}(-R) P_{B_i} \preceq \epsilon P_{B_i}. \end{aligned}$$

It follows that $\|P_{C_{i,j}} R P_{C_{i,j}}\| \leq \epsilon$ for all i, j.

(2) $\Rightarrow$ (6): By (2), for every $\epsilon > 0$, there exists $r \in \mathbb{N}$ such that the following is true. Given Q with $Q = Q^*$, $0 \preceq Q \preceq I$ and $Q_{i,i} \in [0, \delta]$, let $A = Q - E(Q)$. We have $-\delta I \preceq A \preceq I$, so $\|A\| \leq 1$. There exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} A P_{A_i}\| \leq \epsilon\delta$ for all i. Then

$$\|P_{A_i} Q P_{A_i}\| \leq \|P_{A_i} E(Q) P_{A_i}\| + \|P_{A_i} A P_{A_i}\| \leq \delta + \epsilon\delta.$$

(6) $\Rightarrow$ (5): For any $\epsilon > 0$ and even $N \geq 2$, let r be such that (6) is true. Let $Q \in \mathbb{C}^{n \times n}$ satisfy $Q = Q^* = Q^2$ and $E(Q) = I/2$. Let $J \in \mathbb{R}^{N \times N}$ be the matrix of all ones. Let $k = N/2$ and let $M \in \mathbb{C}^{nk \times nk}$ be the Kronecker product $Q \otimes J/k$. It is easy to check that $M = M^* = M^2$ and $M_{i,i} = 1/2k = 1/N$. By (6), there exists a partition $\{A_1, ..., A_r\}$ of $[nk]$ such that $\|P_{A_i} M P_{A_i}\| \leq (1 + \epsilon)/N \ \forall i \in [r]$. Then

$$\left\|P_{A_i \cap [n]} Q P_{A_i \cap [n]}\right\| \leq k \|P_{A_i} M P_{A_i}\| \leq k(1 + \epsilon)/N \leq (1 + \epsilon)/2.$$

(7) $\Rightarrow$ (6): For any $\epsilon > 0$ and even $N \geq 2$, let r be such that (7) is true. Let $Q \in \mathbb{C}^{n \times n}$ satisfy $Q = Q^* = Q^2$ and $Q_{i,i} \in [0, 1/N]$ for all i. Let V be a $d \times n$ matrix with $Q = V^* V$, where d is the rank of Q. Let v_j be the j^{th} column of V, so $v_j^* v_j = Q_{j,j} \leq 1/N$. By (7), there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that, for all $i \in [r]$, we have $\left\|\sum_{j \in A_i} v_j v_j^*\right\| \leq (1 + \epsilon)/N$.

For any complex matrix M, the matrices MM^* and M^*M have the same non-zero eigenvalues. It follows that, for any $A \subseteq [n]$,

$$\|P_A Q P_A\| = \|P_A V^* V P_A\| = \|V P_A V^*\| = \left\|\sum_{j \in A} v_j v_j^*\right\|. \tag{6.1}$$

So, $\|P_{A_i} Q P_{A_i}\| \leq (1 + \epsilon)/N$ for all $i \in [r]$. This proves (6).

(6) $\Rightarrow$ (7): For any $\epsilon > 0$ and even $N \geq 2$, let r be such that (6) is true. Let $v_1, ..., v_n \in \mathbb{C}^d$ satisfy $\sum_j v_j v_j^* = I$ and $v_j^* v_j \leq 1/N$. Define the $n \times n$ matrix Q by $Q_{i,j} = v_i^* v_j$. That is, if $V \in \mathbb{C}^{d \times n}$ is the matrix whose j^{th} column is v_j then $Q = V^* V$. Clearly $Q = Q^*$, and $Q^2 = V^* V V^* V = V^* V = Q$. Furthermore, $Q_{i,i} \in [0, 1/N]$ because $Q_{i,i} = v_i^* v_i$. By (6), there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} Q P_{A_i}\| \leq (1 + \epsilon)/N$ for all $i \in [r]$. By (6.1), this implies (7). ■

6.2 Finite-dimensional paving conjectures with existential quantification

In this section we briefly observe that the conjectures of the previous section are equivalent under replacing the universal quantification "$\forall \epsilon > 0$" with the existential quantification "$\exists \alpha \in (0,1)$".

Theorem 6.4. The following are equivalent:

1: the Kadison-Singer problem has a positive solution,

2: $\exists \alpha \in (0,1)$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ and for every self-adjoint matrix $A \in \mathbb{C}^{n\times n}$ with $E(A) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} A P_{A_i}\| \leq \alpha \|A\|$ $\forall i \in [r]$.

3: $\exists \alpha \in (0,1)$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ and for every $R \in \mathbb{C}^{n\times n}$ with $R = R^* = R^{-1}$ (i.e., a unitary reflection matrix) and $E(R) = 0$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} R P_{A_i}\| \leq \alpha$ $\forall i \in [r]$.

4: $\exists \alpha \in (0,1)$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ and for every $Q \in \mathbb{C}^{n\times n}$ with $Q = Q^* = Q^2$ (i.e., a complex orthogonal projection matrix) and $E(Q) = I/2$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} Q P_{A_i}\| \leq (1+\alpha)/2$ $\forall i \in [r]$.

5: $\exists \alpha \in (0,1)$, $\exists$ **even** $N \geq 2$, $\exists r \in \mathbb{N}$, $\forall n \in \mathbb{N}$ and for every $Q \in \mathbb{C}^{n\times n}$ with $Q = Q^* = Q^2$ and $Q_{i,i} \in [0, 1/N]$ for all i, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that $\|P_{A_i} Q P_{A_i}\| \leq (1+\alpha)/N$ $\forall i \in [r]$.

6: $\exists \alpha \in (0,1)$, $\exists$ **even** $N \geq 2$, $\exists r \in \mathbb{N}$, $\forall d \in \mathbb{N}$ and for all $v_1, ..., v_n \in \mathbb{C}^d$ satisfying $\sum_j v_j v_j^* = I$ and $v_j^* v_j \leq 1/N$ for all $j \in [n]$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that we have $\left\|\sum_{j\in A_i} v_j v_j^*\right\| \leq (1+\alpha)/N$ $\forall i \in [r]$.

Proof. (1) $\Rightarrow$ (2): Follows from Theorem 5.11 and Theorem 6.1.

(2) $\Rightarrow$ (1): Let $\alpha \in (0,1)$ and $r \in \mathbb{N}$ be such that (2) is true. Let $M \in \mathbb{C}^{n\times n}$ be self-adjoint with $E(M) = 0$. Fix any $\epsilon > 0$. By (2), there exists a partition $\mathcal{A}_1$ of $[n]$ such that

$$\|P_A M P_A\| \leq \alpha \|M\| \quad \forall A \in \mathcal{A}_1.$$

Applying this argument separately to each matrix in $\{ P_A M P_A : A \in \mathcal{A}_1 \}$, we obtain a partition $\mathcal{A}_2$ of $[n]$ such that

$$\|P_A M P_A\| \leq \alpha^2 \|M\| \quad \forall A \in \mathcal{A}_2.$$

Repeating $k := \lceil \log_\alpha \epsilon \rceil$ times, we obtain a partition $\mathcal{A}_k$ of $[n]$ such that

$$\|P_A M P_A\| \leq \epsilon \|M\| \quad \forall A \in \mathcal{A}_k.$$

This shows that Theorem 6.1(2) holds, which implies (1) by Theorem 5.11.

The remainder of the equivalences are similar to the proof of Theorem 6.3. ■

7 Weaver's conjecture

Building on work of Akemann and Anderson [AA91], Weaver [Wea04] states a conjecture that appears to be weaker than Theorem 6.3 (7) (or Theorem 6.4 (6)), but is actually equivalent.

Conjecture 7.1 (Conjecture KS_r). $\exists \alpha > 0$, $\exists N \geq 2$, $\forall d \in \mathbb{N}$, and for all $w_1, ..., w_n \in \mathbb{C}^d$ satisfying $\sum_j w_j w_j^* \preceq I$ and $w_j^* w_j \leq 1/N$ for all $j \in [n]$, there exists a partition $\{A_1, ..., A_r\}$ of $[n]$ such that, for all $i \in [r]$, we have $\left\|\sum_{j\in A_i} w_j w_j^*\right\| \leq 1 - \alpha/N$.

Theorem 7.2 ([Wea04, Theorem 1]). The following are equivalent:

1: the Kadison-Singer problem has a positive solution,

2: There exists $r \geq 2$ such that Conjecture KS_r is true.

Proof. (1) $\Rightarrow$ (2): Directly from Theorem 6.3 (7).

(2) $\Rightarrow$ (1): Weaver's proof is based on results of Akemann and Anderson [AA91], which require much deeper operator theory than we are able to discuss here. ■

The recent breakthrough of Marcus, Spielman and Srivastava is as follows:

Theorem 7.3 (Corollary 1.3 in [MSS13]). Let $u_1, ..., u_m \in \mathbb{C}^d$ satisfy $\sum_i u_i u_i^* = I$ and $\|u_i\|_2^2 \leq \delta$ for all i. Then there exists a partition of $\{1, ..., m\}$ into two sets S_1 and S_2 such that

$$\left\| \sum_{i \in S_j} u_i u_i^* \right\|_2 \leq \frac{(1 + \sqrt{2\delta})^2}{2} \qquad \forall j \in \{1, 2\}\,. \tag{7.1}$$

This directly implies Conjecture KS_2, and therefore a positive solution to Problem 1.1.

Question 7.4. Does Theorem 7.3 imply a solution to the Kadison-Singer problem without using KS_r as an intermediate step? For example, does Theorem 7.3 directly imply any of the statements in Theorem 6.3?

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A Point-set Topology

We state some standard definitions and facts from point-set topology. Some references for this material are Folland [Fol99, Ch. 4], Royden [Roy88, Ch. 8], or Wikipedia.

Definition A.1. Let X be any non-empty set. A ***topology*** $\mathcal{T}$ on X is a family of subsets of X such that

- $\emptyset, X \in \mathcal{T}$,
- $\mathcal{T}$ is closed under arbitrary unions, and
- $\mathcal{T}$ is closed under finite intersections.

A pair $(X, \mathcal{T})$ is called a ***topological space*** if $\mathcal{T}$ is a topology on X. When $\mathcal{T}$ is apparent or need not be named, we simply say that X is a topological space. The members of $\mathcal{T}$ are called ***open*** sets. A set $A \subseteq X$ is ***closed*** if A^c is open. A set that is both closed and open is called ***clopen***. In every topological space X, both $\emptyset$ and X are clopen.

For any $S \subseteq X$, the ***interior*** of S, denoted $\mathrm{int}(S)$ is the union of all open sets contained in S. A ***neighborhood*** of $x \in X$ is any set S such that $x \in \mathrm{int}(S)$. Equivalently, N is a neighborhood of x if N contains an open set that contains x.

Definition A.2. Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{T}$ is called a ***base*** for $\mathcal{T}$ (or a ***base for the open sets*** of $\mathcal{T}$) if, for every $x \in X$ and every open set U containing x, there exists $V \in \mathcal{B}$ with $x \in V \subseteq U$.

For any $S \subseteq X$, the ***closure*** of S, denoted $\overline{S}$, is the intersection of all closed sets containing S. A set $S \subseteq X$ is ***dense*** in X if its closure equals X. Equivalently, letting $\mathcal{B}$ be a base, S is dense if every non-empty $B \in \mathcal{B}$ has $B \cap S \neq \emptyset$. A topological space is ***separable*** if it has a countable dense subset. For example, $\mathbb{R}^n$ is separable because $\mathbb{Q}^n$ is a countable dense subset.

Fact A.3. Let X be a non-empty set and $\mathcal{B} \subseteq 2^X$. Suppose that $\bigcup \mathcal{B} = X$ and $\mathcal{B}$ is closed under finite intersections. Let $\mathcal{T} = \left\{ \bigcup_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{B} \right\}$ be the collection of all unions of members of $\mathcal{B}$. Then $\mathcal{T}$ is a topology and $\mathcal{B}$ is a base for $\mathcal{T}$.

Definition A.4. Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{F}$ is called a ***base for the closed sets*** for $\mathcal{T}$ if every closed set is an intersection of members of $\mathcal{F}$.

Fact A.5. Let X be a topological space. A family $\mathcal{F}$ is a base for the closed sets if and only if the family of complements of members of $\mathcal{F}$ is a base for the open sets.

Definition A.6. Let $(X, \mathcal{T})$ be a topological space. Suppose that, for every collection $\{T_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ satisfying $X = \bigcup_{\alpha \in A} T_\alpha$, there exists a finite subcollection $T_1, ..., T_k$ satisfying $X = \bigcup_{i=1}^k T_i$. Then X is called ***compact***.

Definition A.7. Let $\mathcal{C}$ be a collection of sets. Suppose that, for every finite subcollection $C_1, ..., C_k \in \mathcal{C}$, we have $\bigcap_{i=1}^k C_i \neq \emptyset$. Then we say that $\mathcal{C}$ has the ***finite intersection property***.

Claim A.8. Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{F}$ be a base for the closed sets. The following are equivalent.

(i): X is compact.

(ii): For every collection $\{C_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, we have $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$.

(iii): For every subcollection $\{F_\beta\}_{\beta \in B} \subseteq \mathcal{F}$ with the finite intersection property, we have $\bigcap_{\beta \in B} F_\beta \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii): Let $\{C_\alpha\}_{\alpha \in A}$ be a collection of closed sets with the finite intersection property. Define the collection $\mathcal{D} = \{D_\alpha\}_{\alpha \in A}$ of open sets by $D_\alpha = C_\alpha^c$. Then, for every finite subcollection $D_1, ..., D_k \in \mathcal{D}$, we have $\bigcup_{i=1}^k D_i = (\bigcap_{i=1}^k C_i)^c \neq (\emptyset)^c = X$. By (i), we must have $X \neq \bigcup_{\alpha \in A} D_\alpha$, so $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (i): Let $\{T_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ satisfy $X = \bigcup_{\alpha \in A} T_\alpha$. Then $C_\alpha := T_\alpha^c$ is closed and $\bigcap_{\alpha \in A} C_\alpha = \emptyset$. Each C_α can be written $C_\alpha = \bigcap_{F \in \mathcal{F}_\alpha} F$ for some subcollection $\mathcal{F}_\alpha \subseteq \mathcal{F}$. So $\bigcap_{F \in \cup_\alpha \mathcal{F}_\alpha} F = \emptyset$. By (iii), $\bigcup_\alpha \mathcal{F}_\alpha$ does not have the finite intersection property. That is, there exists sets $F_1, ..., F_k \in \bigcup_\alpha \mathcal{F}_\alpha$ with $\bigcap_{i=1}^k F_i = \emptyset$. For $i = 1, ..., k$, choose $\alpha_i \in A$ satisfying $C_{\alpha_i} \subseteq F_i$. Then $\bigcap_{i=1}^k C_{\alpha_i} = \emptyset$ also, so $\bigcup_{i=1}^k T_{\alpha_i} = X$. ∎

Definition A.9. Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if $f^{-1}(V)$ is open for every open $V \subseteq Y$.

Fact A.10. Let X and Y be topological spaces where X is compact. If $f : X \to Y$ is continuous then $f(X)$ is compact.

Fact A.11. $f : X \to Y$ is continuous if and only if, for every $x \in X$ and for every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.

A topology on X is ***Hausdorff*** if, for every distinct $x, y \in X$, there exist disjoint open sets S_x and S_y with $x \in S_x$ and $y \in S_y$. For example, for finite n, the standard topology on $\mathbb{R}^n$ or $\mathbb{C}^n$ is Hausdorff.

Fact A.12. Let X and Y be topological spaces, where Y is Hausdorff. Let $f, g : X \to Y$ be continuous functions that agree on a dense subset of X. Then f and g agree on all of X.

B Banach and Hilbert Spaces

We state some standard definitions and facts from functional analysis. Some references for this material are Albiac and Kalton [AK06, Ch. 1, 2], Folland [Fol99, Ch. 5, 6], Heil [Hei10, Ch. 4], Kadison-Ringrose [KR83, Ch. 1, 2], and Rudin [Rud73].

Definition B.1. A ***Banach space*** is a (real or) complex vector space equipped with a norm for which the space is complete (i.e., every Cauchy sequence converges) with respect to the norm metric.

Some important Banach spaces, that we will discuss further in Appendix B.1, are the ***sequence spaces*** ℓ_p. These are Banach spaces for $p \in [1, \infty]$.

Example B.2. Let X be a topological space. Let $C(X)$ be the set of all bounded, continuous functions $f : X \to \mathbb{C}$. This is a Banach space with the ***supremum norm***

$$\|f\|_u \quad = \quad \sup \{ \, |f(x)| \, : \, x \in X \, \} \, .$$

If X is compact then, by Fact A.10, the word "bounded" is superfluous: $C(X)$ contains all continuous functions $f : X \to \mathbb{C}$. ∎

In a topological vector space, such as a Banach space, we can discuss bases that involve infinite sums.

Definition B.3. A ***Schauder basis*** in a Banach space is a countable sequence $E = (x_1, x_2, ...)$ such that $\|x_i\| = 1$ for all i and if every point x in the vector space can be *uniquely* written as $x = \sum_i \alpha_i x_i$.

The convergence of this countably infinite sum is with respect to the norm of the Banach space. The sum might be only conditionally convergent, which is why the ordering of the basis is important. The Schauder basis E is called ***unconditional*** if, whenever $\sum_i \alpha_i x_i$ converges, it converges unconditionally.

Remark B.4. Only separable Banach spaces can have a Schauder basis.

Definition B.5. A ***Hilbert space*** is a (real or) complex vector space equipped with an inner product for which the space is complete with respect to the norm induced by that inner product.

Definition B.6. The ***closed linear span*** of a set E of vectors is, equivalently:

- the closure of the set of all finite linear combinations of elements of E,
- the intersection of all closed linear subspaces that contain E,
- the closure of the intersection of all linear subspaces that contain E.

In a Hilbert space, we can define an orthonormal basis.

Definition B.7. In a Hilbert space H, an ***orthonormal basis*** is a set E such that

- $\|x\| = 1$ for all $x \in E$,
- $\langle x, y \rangle = 0$ for all distinct $x, y \in E$,
- H is the closed linear span of E.

B.1 Sequence Spaces

Definition B.8. A ***sequence space*** is a vector space whose elements are infinite sequences of complex numbers. That is, each element is a function $f : \mathbb{N} \to \mathbb{C}$.

Example B.9. For any $p \in [1, \infty)$, the space ℓ_p is the sequence space consisting of all sequences $x = (x_1, x_2, ...)$ for which $\sum_n |x_n|^p < \infty$. Its norm is $\|x\|_p = (\sum_n |x_n|^p)^{1/p}$. This is a Banach space, and it is separable.

Define the sequence $E = (e_1, e_2, ...)$, where the vector e_i has ith coordinate equal to 1 and all other coordinates equal to 0. E is an unconditional Schauder basis for ℓ_p, for all $p \in [1, \infty)$. ■

Example B.10. The sequence space ℓ_2 together with the inner product $\langle z, w \rangle = \sum_n z_n \overline{w_n}$ is a Hilbert space. The standard Schauder basis is an orthonormal basis. ■

Example B.11. The space ℓ_∞ is the sequence space consisting of all sequences $x = (x_1, x_2, ...)$ for which $\sup_n |x_n| < \infty$. Its norm is $\|x\|_\infty = \sup_n |x_n|$. This is also a Banach space. ■

Claim B.12. ℓ_∞ is not separable.

Proof. Consider the set $S \subset \ell_\infty$ consisting of all sequences of zeros and ones. Note that S is uncountable (there is a surjection to the interval $[0, 1]$). The distance between any two distinct elements of S is 1. So the balls $\{ B(x, 1/3) : x \in S \}$ are disjoint. But any dense set must contain at least one element from each ball, so any dense set must be uncountable. ■

Because ℓ_∞ is not separable, it cannot have a Schauder basis. To illustrate the issue, note that the all-ones vector is in ℓ_∞, but the sum $\sum_{i\geq 1} e_i$ does not converge (in the ℓ_∞ norm), so E is not a Schauder basis.

B.2 Operators on ℓ_2

Definition B.13. Let X and Y be normed spaces. A ***bounded linear operator*** $L : X \to Y$ is a linear map for which there exists a real number $M > 0$ for which $\|Lv\|_Y \leq M \|v\|_X$ for all $v \in X$. The ***operator norm*** of L is the smallest such number M.

The algebra of bounded linear operators on ℓ_2 is denoted $B(\ell_2)$.

Example B.14. A **diagonal operator** on ℓ_2 is a linear map $L : \ell_2 \to \ell_2$ for which there exist complex numbers $c_1, c_2, ...$ such that

$$L\big((v_1, v_2, ...)\big) \;=\; (c_1 v_1, c_2 v_2, ...)$$

for all $v = (v_1, v_2, ...) \in \ell_2$.

The map L is a ***bounded diagonal operator*** if

$$\sup\left\{ \sum_i |c_i|^2 |v_i|^2 \,:\, \sum_i |v_i|^2 = 1 \right\} \;<\; \infty.$$

Each term in this sup is a convex combination of $\left\{|c_i|^2\right\}_{i\in\mathbb{N}}$, so the sup equals $\sup_i |c_i|^2$. Thus the space of bounded diagonal operators on ℓ_2 is identified with the sequence space ℓ_∞, and the operator norm is identified with $\|\cdot\|_\infty$. The space of all bounded diagonal operators on ℓ_2 is denoted $\mathbb{D}(\ell_2)$. We have argued that $\mathbb{D}(\ell_2)$ and ℓ_∞ are isometrically isomorphic. ■

Definition B.15. A ***projection*** (more accurately, ***orthogonal projection***) in $\mathbb{D}(\ell_2)$ is an operator of the form P_A for $A \subseteq \mathbb{N}$, where $\langle\, P_\sigma e_i,\, e_i \,\rangle$ is 1 if $e_i \in A$ and otherwise 0.

Fact B.16. The linear span of the projections in $\mathbb{D}(\ell_2)$ is dense in $\mathbb{D}(\ell_2)$.

Proof. Fix an operator $D \in \mathbb{D}(\ell_2)$. First assume $D \geq 0$. For $n \geq 1$ and $i \in \left\{0, ..., 2^{2n}\right\}$, define $A_i = \{\, j \,:\, \lfloor 2^n \langle\, De_j,\, e_j \,\rangle \rfloor = i \,\}$. Let $D_n = 2^{-n} \sum_{i=0}^{2^{2n}} P_{A_i}$. Then $D_n \to D$ as $n \to \infty$. For the case of arbitrary D, simply write $D = D^+ - D^-$ where $D^+, D^- \geq 0$ and apply the same argument. ■

C Operator Theory

We state some standard definitions and facts from operator theory. Some references for this material are Kadison-Ringrose [KR83], Rudin [Rud73, Ch. 10, 12].

Definition C.1. A ***Banach algebra*** $\mathcal{A}$ is an associative algebra over $\mathbb{C}$ that is also a Banach space with a norm $\|\cdot\|$. The norm must be submultiplicative, i.e., $\|xy\| \leq \|x\|\,\|y\|$ for all $x, y \in \mathcal{A}$.

A Banach algebra is called ***unital*** if it contains the multiplicative identity.

Example C.2. The algebra $\mathbb{C}^{n\times n}$ of $n \times n$ matrices over $\mathbb{C}$, together with a submultiplicative norm (such as the induced 2-norm or the Frobenius norm), is a unital Banach algebra. ■

Definition C.3. An ***involutive Banach algebra*** is a Banach algebra $\mathcal{A}$ over $\mathbb{C}$, together with a map $* : \mathcal{A} \to \mathcal{A}$ such that

- $(x^*)^* = x$ for all $x \in \mathcal{A}$ (i.e., $*$ is an involution),
- $(x+y)^* = x^* + y^*$ for all $x, y \in \mathcal{A}$,
- $(xy)^* = y^* x^*$ for all $x, y \in \mathcal{A}$,
- $(\lambda x)^* = \overline{\lambda} x^*$ for all $x \in \mathcal{A}$, $\lambda \in \mathbb{C}$,
- $\|x\| = \|x^*\|$ for all $x \in \mathcal{A}$ (i.e., $*$ is an isometry).

Definition C.4. A ***C^*-algebra*** is an involutive Banach algebra for which the $*$-involution satisfies $\|xx^*\| = \|x\|^2$.

Example C.5. Consider again $\mathbb{C}^{n \times n}$, using the induced 2-norm as a norm, and using conjugate transpose (i.e., adjoint) as the involution $*$. This is a C^*-algebra. ■

Example C.6. Let X be a compact Hausdorff topological space and, following Example B.2, let $C(X)$ be the Banach space of complex-valued continuous functions on X. Define a multiplication operation on $C(X)$ by pointwise multiplication, and define the involution $*$ by pointwise conjugation. Then $C(X)$ becomes a C^*-algebra. ■

Example C.7. Consider the Banach space ℓ_∞. Define a multiplication operation on ℓ_∞ by pointwise multiplication, and define the involution $*$ by pointwise conjugation. Then ℓ_∞ becomes a C^*-algebra. ■

Definition C.8. Let $\mathcal{A}$ be a C^*-algebra. An element $A \in \mathcal{A}$ is called ***self-adjoint*** if $A = A^*$.

Definition C.9. Let $\mathcal{A}$ be a C^*-algebra. An element $A \in \mathcal{A}$ is called ***positive*** if $A = B^*B$ for some $B \in \mathcal{A}$. Note that if A is positive, it is also self-adjoint. The notion of positivity gives rise to a partial ordering on self-adjoint elements: for self-adjoint A and B, we write

$$A \succeq B \quad \text{if} \quad A - B \text{ is positive.}$$

References: Kadison-Ringrose [KR83, Theorem 4.2.6], Wikipedia.

Example C.10. Consider a complex Hilbert space H and the algebra $B(H)$ of bounded linear operators on H. If $A \in B(H)$ is positive then $\langle Ax, x \rangle \geq 0$ for all $x \in H$. The converse is also true (assuming that H is complex): if $\langle Ax, x \rangle \geq 0$ for all $x \in H$, then there exists $B \in B(H)$ such that $A = B^*B$. ■

References: Kadison-Ringrose [KR83, pp. 103], Wikipedia.

Let $\mathcal{A}$ be an algebra. A linear functional $f : \mathcal{A} \to \mathbb{C}$ is called ***multiplicative*** if $f(xy) = f(x) \cdot f(y)$ for all $x, y \in \mathcal{A}$.

Definition C.11. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with involution. A mapping $\phi : \mathcal{A} \to \mathcal{B}$ is called a ****-homomorphism*** if it is linear, multiplicative and unital (i.e., $\phi(1) = 1$), and it commutes with conjugation (i.e., $\phi(A^*) = \phi(A)^*$).

Definition C.12. An ***operator space*** is a closed subspace of a C^*-algebra. An ***operator system*** is a self-adjoint subspace of a unital C^*-algebra $\mathcal{A}$ which contains the unit of $\mathcal{A}$.

References: Blackadar [Bla06, pp. 106].

Claim C.13. Let $\mathcal{A}$ be a C^*-algebra and let $f : \mathcal{A} \to \mathbb{C}$ be a linear functional. Then f is completely determined by its values on the self-adjoint elements of $\mathcal{A}$.

Proof. Let $H \in \mathcal{A}$ be arbitrary. Then $H = P + iQ$ where $P = (H + H^*)/2$ and $Q = i(H^* - H)/2$ are both self-adjoint. By linearity, $f(H) = f(P) + i\, f(Q)$. ■

C.1 States

Definition C.14. Let $\mathcal{A}$ be a C^*-algebra. A ***positive linear functional*** on $\mathcal{A}$ is a linear map $f : \mathcal{A} \to \mathbb{C}$ for which $f(A) \geq 0$ for all *positive* elements $A \in \mathcal{A}$. If additionally $\mathcal{A}$ is unital and f is unital (i.e., $f(1) = 1$) then f is called a ***state***.

References: Arveson [Arv76, pp. 27], Kadison-Ringrose [KR83, pp. 255].

Claim C.15. Let $\mathcal{A}$ be a C^*-algebra and let $f : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Then f respects the $\preceq$ ordering on $\mathcal{A}$, i.e., $A \preceq B$ implies $f(A) \preceq f(B)$.

Definition C.16. A linear functional f on $\mathcal{A}$ is called ***Hermitian*** if $f(A^*) = \overline{f(A)}$ for all $A \in \mathcal{A}$.

Claim C.17. Let $\mathcal{A}$ be a C^*-algebra and let $f : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Then f is Hermitian.

Proof. Let $H \in \mathcal{A}$ be self-adjoint. We may write $H = P - Q$ where $P := (\|H\|\, I + H)/2$ and $Q := (\|H\|\, I - H)/2$. We claim that $P, Q \succeq 0$, which implies that $f(P), f(Q) \geq 0$. Then $f(H) = f(P) - f(Q)$ is real. That is, $f(H) = \overline{f(H)}$ for all self-adjoint $H \in \mathcal{A}$.

Now let $A \in \mathcal{A}$ be arbitrary. Write $A = P + iQ$ where $P = (A + A^*)/2$ and $Q = i(A^* - A)/2$ are both self-adjoint. So

$$f(A^*) = f(P - iQ) = f(P) - i\, f(Q) = \overline{f(P)} - i\, \overline{f(Q)} = \overline{f(P) + i\, f(Q)} = \overline{f(P + iQ)} = \overline{f(A)},$$

as required. ■

References: Kadison-Ringrose [KR83, pp. 255].

Claim C.18. Let $\mathcal{A}$ be a C^*-algebra and let $f : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Then $\langle\, A,\, B \,\rangle := f(B^* A)$ is an inner product.

References: Kadison-Ringrose [KR83, pp. 256].

Corollary C.19. Let $\mathcal{A}$ be a C^*-algebra and let $f : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Then $|f(B^* A)|^2 \leq f(A^* A) \cdot f(B^* B)$ for all $A, B \in \mathcal{A}$.

Fact C.20. Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{X}$ be an operator system in $\mathcal{A}$, and let ϕ be a state on $\mathcal{X}$. Then ϕ can be extended to a state on $\mathcal{A}$.

References: Blackadar [Bla06, II.6.3.1].

Definition C.21. Let A be a C^*-algebra with unit I. If f is a state that cannot be written as a non-trivial convex combination of two different states, then f is called a ***pure state***. (In other words, the pure states are the extreme points of the convex set of states.)

References: Kadison-Ringrose [KR83, pp. 213].

Example C.22. For any unit vector $x \in \ell_2$, define the functional $\omega_x : B(\ell_2) \to \mathbb{R}$ by $\omega_x(T) = \langle\, Tx,\, x \,\rangle$. It is easy to see that this is a state on $B(\ell_2)$. It is also a pure state on $B(\ell_2)$. ■

References: Arveson [Arv76, Exercise 1.6.A], Kadison-Ringrose [KR83, pp. 256 and Exercise 4.6.68].

Fact C.23. Let X be a compact Hausdorff space and let $C(X)$ be the C^*-algebra defined in Example C.6. A (non-zero) linear functional on $C(X)$ is a pure state if and only if it is multiplicative.

References: Kadison-Ringrose [KR83, Theorem 3.4.7 and Proposition 4.4.1], Rudin [Rud73, Theorem 11.32].

Alternatively, if $\mathcal{A}$ is Abelian then the pure states are again the multiplicative functionals [Dix77, 2.5.2] [KR83, Proposition 4.4.1].

References

[AA91] Charles A. Akemann and Joel Anderson. *Lyapunov Theorems for Operator Algebras.* American Mathematical Society, 1991.

[AK06] Fernando Albiac and Nigel J. Kalton. *Topics in Banach Space Theory.* Springer-Verlag, 2006.

[Arv76] William Arveson. *An Invitation to C^*-Algebras.* Springer-Verlag, 1976.

[Bla06] Bruce Blackadar. *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras.* Springer-Verlag, 2006.

[CEKP07] Pete Casazza, Dan Edidin, Deepti Kalra, and Vern I. Paulsen. Projections and the Kadison-Singer problem. *Operators and Matrices*, 1(3):391–408, 2007.

[CFTW06] Peter G. Casazza, Matthew Fickus, Janet C. Tremain, and Eric Weber. The Kadison-Singer problem in mathematics and engineering: a detailed account. In D. Han, P. E. T. Jorgensen, and D. R. Larson, editors, *Contemporary Mathematics*, volume 414, pages 299–356. 2006.

[Dix77] Jacques Dixmier. *C^*-algebras.* North Holland, 1977.

[Fol99] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications.* Wiley-Interscience, second edition, 1999.

[Hei10] Christopher Heil. *A Basis Theory Primer.* Birkhäuser, 2010.

[Hin96] Neil Hindman. Algebra in βS and its applications to Ramsey theory. *Japonica*, 44:581–625, 1996.

[HKW87] H. Halpern, V. Kaftal, and G. Weiss. Matrix pavings in $B(H)$. *Operator Theory: Advances and Applications*, 24:201–214, 1987.

[HS98] Neil Hindman and Dona Strauss. *Algebra in the Stone-Čech Compactification.* De Gruyter, 1998.

[KR83] Richard V. Kadison and John R. Ringrose. *Fundamentals of the Theory of Operator Algebras.* American Mathematical Society, 1983.

[Kre40] Mark G. Krein. Sur le problème de prolongement des functions hermitiennes positives et continues. *Dokl. Acad. Nauk. SSSR*, 26:17–22, 1940.

[KS59] Richard V. Kadison and I. M. Singer. Extensions of pure states. *Americal Journal of Mathematics*, 81(2):383–400, 1959.

[MSS13] Adam Marcus, Daniel A. Spielman, and Nikhil Srivastava. Interlacing Families II: Mixed Characteristic Polynomials and The Kadison-Singer Problem, 2013. `http://arxiv.org/abs/1306.3969`.

[PR08] Vern I. Paulsen and Mrinal Raghupathi. Some new equivalences of Anderson's paving conjectures. *Proceedings of the American Mathematical Society*, 136(12):4275–4282, 2008.

[Roy88] H. L. Royden. *Real Analysis*. Macmillan, 1988.

[Rud73] Walter Rudin. *Functional Analysis*. McGraw-Hill, 1973.

[Rud76] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, third edition, 1976.

[Tan91] Betul Tanbay. Pure state extensions and compressibility of the l_1-algebra. *Proceedings of the American Mathematical Society*, 113(3):707–713, 1991.

[Wat11] John Watrous. Theory of Quantum Information, 2011. `http://cs.uwaterloo.ca/~watrous/CS766/`.

[Wea04] Nik Weaver. The Kadison-Singer problem in discrepancy theory. *Discrete Mathematics*, 278:227–239, 2004.