UBC CPSC 536N: Sparse Approximations	Winter 2013
Lecture 9 — January 30, 2013	
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1 Transshipments

Let $G = (V, A)$ be digraph. Given G and capacities for the weights $c : A \to \mathbb{R}$, recall that a $s-t$ flow on G is a function $f : A \to \mathbb{R}$ such that

1. (Conservation of flow) Except for two distinguished nodes, $s, t \in V$ (the source and sink respectively), $f(\delta^- v) - f(\delta^+ v) = 0$.
2. (Capacity constraints) $0 \le f \le c$.

A $b-$transshipment is a generalization of $s-t$ flows, where the requirement that there are two distinguished nodes, s, t is relaxed. Instead, we allow every node to be either a source or a sink. Let the *excess* of $f : A \to \mathbb{R}$ be

$$\text{excess}_f : V \to \mathbb{R} : v \mapsto f(\delta^- \{v\}) - f(\delta^+ \{v\}).$$

We say that f is a $b-$*transshipment* if for $b : V \to \mathbb{R}$.

$$\text{excess}_f = b.$$

Remark 1. (Interpretation as oil transport) It is instructive to interpret the graph $G = (V, A)$ as an system of oil transport, with the edges representing oil pipelines between towns. Let f be a $b-$transshipment. b will be the surplus or deficit generated by each town, i.e. if $b\{v\} > 0$, v is a consumer (because more oil is coming in than going out) and if $b\{v\} < 0$, v is a producer (because more oil is coming out than going in). A $b-$transshipment therefore, can be seen as a system of transport which ensures there is no oil buildup or deficit in any node.

All $b-$transshipments are characterized in the following theorem.

Theorem 2. *(Gale's Theorem) Let $D = (V, A)$ be a digraph and $c \in A \to \mathbb{R}$ be the capacities for each edge. For $b \in V \to \mathbb{R}$, there exists a $b-$transshipment f with $0 \le f \le c$ iff*

$$-b(U) \le c(\delta^+ U), \qquad \forall U \subseteq V \tag{1.1}$$
$$b(V) = 0. \tag{1.2}$$

Furthermore, if b and c are integral, then f can be chosen to be integral. Furthermore, given any weights w on the arcs, we can compute such an f of minimum weight in polynomial time.

Proof Sketch: We reduce the problem of finding a $b-$transshipment to the problem of finding an $s-t$ flow.

Before embarking on the proof, we first need to construct a graph, $D' = (V, A')$ with capacities $c' : A' \to \mathbb{R}$. Let

$$P = \{v : b\{v\} < 0\} \qquad \text{(producer)}$$
$$C = \{v : b\{v\} \ge 0\} \qquad \text{(consumer)}$$

Construct the graph as follows.

1. $\forall a \in A$, a is also an arc of D' with capacity $c'(a) = c(a)$.
2. $\forall v \in P$, add an arc (s, v) with capacity $c'(s, v) = -b(v)$. (The interpretation is that v is a "producer"; the oil that it "produces" actually comes from s.)
3. $\forall v \in C$, add an arc (v, t) with capacity $c'(v, t) = b(v)$. (The interpretation is that v is a consumer; the oil that it "consumes" is actually sent to t.)

Lemma 3. *Assume $b(V) = 0$. The following statements are equivalent.*

1. D' has a flow of $b(C)$.
2. $c'(\delta^+(U \cup \{s\})) \geq b(C)$ for all $U \subseteq V$.
3. $c(\delta^+ U) \geq -b(U)$ for all $U \subseteq V$.

Proof. 1 $\iff$ 2. Observe that $b(C)$ is a lower bound on all s-t cuts. The max-flow min-cut theorem states therefore that the maximum s-t flow is achieved with flow at least $b(C)$. The converse is identical.

2 $\iff$ 3. By the definition of D' we have

$$c'(\delta^+(U \cup \{s\})) = c(\delta^+ U) - b(P \cap \bar{U}) + b(C \cap U).$$

So, using the identities $b(C) = b(C \cap \bar{U}) + b(C \cap U)$, $\bar{P} = C$ and $b(U) = -b(\bar{U})$ we have

$$\begin{aligned}
c'(\delta^+(U \cup \{s\})) - b(C) &= c(\delta^+ U) - b(P \cap \bar{U}) + b(C \cap U) - b(C) \\
&= c(\delta^+ U) - b(P \cap \bar{U}) + b(C \cap U) - \left[b(C \cap \bar{U}) + b(C \cap U)\right] \\
&= c(\delta^+ U) - [b(P \cap \bar{U}) + b(C \cap \bar{U})] \\
&= c(\delta^+ U) - b(\bar{U}) \\
&= c(\delta^+ U) + b(U)
\end{aligned}$$

Therefore $c'(\delta^+(U \cup \{s\})) - b(C) \geq 0$ iff $c(\delta^+ U) + b(U) \geq 0$. ☐

We proceed to prove Theorem 2.

Proof. Obviously if $b(V) \neq 0$ then D cannot have a b-transshipment, so assume $b(V) = 0$. We will show that

$$D \text{ has a } b\text{-transshipment } f \text{ such that } 0 \leq f \leq c \quad \iff \quad D' \text{ has a flow } f' \text{ of value } b(C). \tag{1.3}$$

Then (ignoring computability and integrality), the result follows by the equivalence of 1 and 3 in Lemma 3.

We use the following mapping between and transshipment f in D and a flow f' in D'.

- $f(a) = f'(a)$ for all arcs in A
- $f'(a) = c'(a)$ for arcs incident on s or t

Obviously $0 \leq f \leq c$ iff $0 \leq f' \leq c'$, and if f' is a flow, it must have value $b(C)$. For all $v \in V$ we have

$$\text{excess}_{f'}(v) = \text{excess}_f(v) - b(v)$$

because for $v \in P$ the sv arc contributes $-b(v)$ and for $v \in C$, the vt arc contributes $-b(v)$. Thus, f' is a flow iff f is a b-transshipment (which means $\text{excess}_f = b$). This proves (1.3).

If b and c are integral then c' is integral so, by the max-flow min-cut theorem, we can find an integral f', which gives an integral f. Since a minimum weight s-t flow f' of value $b(C)$ can be found in polynomial time, a minimum weight b-transshipment f can also be found in polynomial time. ☐

Remark 4. (Interpretation of Gale's Theorem) Using the metaphor of oil transport, Gale's Theorem states that for any subset of towns, the capacity of the outgoing pipelines must be at least the net amount that those towns produce, and that the entire system has no net surplus/deficit of oil.

Remark 5. (Symmetry in Gale's Theorem) By applying Gale's Theorem to D^{-1} where as D^{-1} is the graph with all arcs reversed, and $-c$, it is easy to see that 1.1 can be replaced by $c(\delta^{-}U) \geq b(U)$.

2 Randomized Rounding for the ATSP

We now return to the analysis of the $O(\log n)$ approximation algorithm for the ATSP. We follow the notation of Lecture 8 (a recap of the notation can be found in the appendix), however for tidiness we omit parenthesis enclosing a function argument when there is no ambiguity. Also note that any function from a discrete set to the reals can be identified with a real valued vector given an ordering of the elements. Both notions will be used interchangeably.

2.1 Controlling All Cuts

Recall that in the previous lecture we proved the following claim bounding the deviation of each cuts from the mean

Claim 6. *(Bounding A Cut, Claim 2.4 of Lecture 8) Let U be a cut and $\mu(U) := \mathbf{E}\,|\delta^{+}_{H}U| = k \cdot x(\delta^{+}U)$. Then*

$$\Pr\left(|\delta^{+}_{H}U| \notin [\tfrac{3}{4}, \tfrac{5}{4}] \cdot \mu(U)\right) \leq 2\exp\left(-\tfrac{1}{48}\mu\right)$$

In the following claim we will show this property is uniform on all cuts in the graph.

Claim 7. *(Bounding All Cuts) Let $\mu(U) = \mathbf{E}\,\delta^{+}_{H}(U)$. Then*

$$\Pr\left(\text{every cut } U \text{ satisfies } \tfrac{3}{4}\mu(U) \leq |\delta^{+}_{H}U| \leq \tfrac{5}{4}\mu(U)\right) \;\geq\; 1 - \tfrac{3}{n}. \tag{2.1}$$

Proof Sketch: We union bound over all possible subsets of U by observing that there are few small cuts, and the large cuts have small probability of violating the inequality.

Proof. Let (U_i) be an ordering of the cuts such that $1 \leq x(\delta^{+}U_1) \leq x(\delta^{+}U_2) \leq \cdots$. Let $\mathcal{E}_i$ be the event

$$\mathcal{E}_i := \left\{\; |\delta^{+}_{H}U_i| \notin [\tfrac{3}{4}, \tfrac{5}{4}] \cdot \mu(U_i) \;\right\}.$$

Then

$$\begin{aligned}
\Pr\left(\textstyle\bigcup_i \mathcal{E}_i\right) &\overset{(1)}{\leq} \textstyle\sum_i \Pr(\mathcal{E}_i) \\
&\overset{(2)}{\leq} \textstyle\sum_i 2\exp(-k \cdot x(\delta^{+}U_i)/48) \\
&\overset{(3)}{\leq} \textstyle\sum_i 2\exp(-4\ln(n) \cdot x(\delta^{+}U_i)) \\
&\leq \textstyle\sum_i \exp(-3\ln(n) \cdot x(\delta^{+}U_i)).
\end{aligned}$$

(1) follows from the union bound (or the sub-additivity of measure), (2) uses Claim 6 and the fact that $\mathbf{E}\,|\delta^{+}_{H}U_i| = k \cdot x(\delta^{+}U_i)$, and (3) uses the definition of k. We now split the sum into two pieces and bound each part separately.

Piece 1 *"Small Cuts". By the strongly connected property of of the* (LP), $x(\delta^{+}U_i) \geq 1$. Therefore

$$\begin{aligned}
\textstyle\sum_{i=1}^{n^2} \Pr(\mathcal{E}_i) &\leq \textstyle\sum_{i=1}^{n^2} \exp(-3\ln(n) \cdot x(\delta^{+}U_i)) \\
&\leq \textstyle\sum_{i=1}^{n^2} n^{-3} = 1/n.
\end{aligned}$$

Piece 2 *"Large Cuts"*. Applying (Corollary 1.5 of Lecture 8) with $\alpha = \ln(i)/2\ln(n)$ (observe that $\alpha \geq 1$), we have $|\{j : x(\delta^+ U_j) \leq \alpha\}| < n^{2\alpha} = i$. The i^{th} cut in the ordering must have $x(\delta^+ U_i) > \alpha$. Therefore

$$\begin{aligned}
\textstyle\sum_{i>n^2} \Pr(\mathcal{E}_i) &\leq \textstyle\sum_{i>n^2} \exp(-3\ln(n) \cdot x(\delta^+ U_i)) \\
&\leq \textstyle\sum_{i>n^2} \exp(-3\ln(n) \cdot \alpha) \\
&= \textstyle\sum_{i>n^2} i^{-3/2} \quad \text{(plugging in value of } \alpha) \\
&< \textstyle\int_{n^2}^{\infty} x^{-3/2} dx = 2/n.
\end{aligned}$$

Combining the two pieces, we get $3/n$. □

2.2 Patch Up

So, with probability at least $0.9 - 3/n$, the graph H:

1. *Has all cuts near their expectations:* By Claim 7, with probability at least $1 - 3/n$, the graph H satisfies satisfies $\frac{3}{4}\mu(U) \leq |\delta_H^+(U)| \leq \frac{5}{4}\mu(U)$ for every $U \subseteq V$.

2. *Is strongly connected:* A consequence of the previous inequality is that $|\delta_H^+(U)| \geq 1$, since $\mu(U) = K \cdot x(\delta^+(U)) \geq 200$.

3. *Has low weight:* By Claim 2.2 in the Lecture 8 notes, with probability at least 0.9 the weight of H is $O(\log n) \cdot w^{\mathsf{T}} x$.

Despite H's desirable properties, it is generally not Eulerian. Fortunately, H is not far from being Eulerian. To establish this, we will find a graph F (in polynomial time) such that

1. $H + F$ is Eulerian, and

2. the weight of F is $O(\log n) \cdot w^T x$.

The Eulerian condition requires that for $v \in V$,

$$0 = |\delta^-_{H+F} v| - |\delta^+_{H+F} v| = \text{excess}_{H+F} v = \text{excess}_H v + \text{excess}_F v.$$

Define $b : V \to \mathbb{R}$ by $b = -\text{excess}_H$. Therefore $H + F$ is Eulerian iff F is a b-transshipment. To prove the existence and computability of F, we invoke Gale's Theorem.

Claim 8. *Consider the b-transshipment problem with infinite capacities on the arcs. With high probability (over the random choice of H) there exists F, an integral b-transshipment, with weight $O(\log n) \cdot w^T x$. Furthermore such an F is computable in polynomial time.*

Proof. We will show that there is a *feasible* solution (not necessarily optimal or integral) with weight $O(\log n) \cdot w^T x$. Consider the b-transshipment with capacities $c = K \cdot x$. (Note that these capacities are not necessarily integral.) We may apply Theorem 2 because

$$\begin{aligned}
-b(U) &= \text{excess}_H(U) \\
&= |\delta^-_H U| - |\delta^+_H U| \\
&\overset{1}{\leq} K \cdot \left(\tfrac{5}{4} x(\delta^- U) - \tfrac{3}{4} x(\delta^+ U)\right) \\
&\overset{2}{=} K \cdot x(\delta^+ U)/2 \leq c(\delta^+ U).
\end{aligned}$$

(1) comes from the upper bound in Claim 7 (which occurs with probability at least $1 - 3/n$). (2) follows from fractional Eulerian property of x.

Therefore there exists a b-transshipment f with $0 \le f \le c$. The weight of f is

$$\sum_{a\in A} w(a)\cdot f(a) \;\le\; \sum_{a\in A} w(a)\cdot c(a) \;=\; K\cdot \sum_{a\in A} w(a)\cdot x(a) \;=\; K\cdot w^{\mathsf{T}}x.$$

Now let the capacities be infinite. Since b is integral, there exists a minimum weight b-transshipment F that is also integral. Furthermore, since f is still feasible with the infinite capacities, the weight of F is at most the weight of f, which is $K\cdot w^{\mathsf{T}}x$. As observed in Theorem 2, such an F can be computed in polynomial time. □

To summarize, we have found integral graphs H and F such that, with probability at least $0.9 - 3/n$,

1. $H + F$ is Eulerian,
2. the weight of $H + F$ is $O(\log n)\cdot w^{\mathsf{T}}x$, and

By the shortcutting argument given at the end of Lecture 6, we can find a Hamilton cycle with weight at most $O(\log n)\cdot w^{\mathsf{T}}x$. This gives a $O(\log n)$ approximation for the ATSP problem.

A Recap of Notation

Notation	**Meaning**
$\delta^+U = \{(v,w) : v\in U, w\notin U\}$	Directed Edges Leaving U
$\delta^-U = \{(w,v) : v\in U, w\notin U\}$	Directed Edges Entering U
$\delta^+_H U = (\delta^+U)\cap H$	Directed Edges Leaving U in the subgraph H

A.1 Algorithm Specific Notation

Notation	**Meaning**
$K = 200\cdot \ln n$	Number of times edges are sampled (a constant)
x	Optimal Solution to LP in Section 1.2 of Lecture 8