UBC CPSC 536N: Sparse Approximations Winter 2013

Lecture 8 — January 28, 2013

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It is important to recall a property we had in a previous lecture, which was

$\forall G$ an undirected graph, $\alpha \in [1, \infty)$ the number of α min cuts of G is $< n^{2\alpha}/2$.

The same result is true if G is a multigraph. Furthermore, the same result is true if G has any non-negative, rational weights — we can scale the weights up to be integers, then view those integral weights as indicating the number of parallel copies of an edge. Lastly, the same result is true if G has any non-negative, real weights because the weights can be approximated arbitrarily well by rationals.

1 LP Formulation of the ATSP

Definition 1.1 (Asymmetric Travelling Salesman Problem, ATSP). Given a directed graph $G = (V, A)$, with weights $w : A \to \mathbb{R}_+$ we want to find

$$\arg\min_{\pi} \left\{ \sum_{v \in V} w_{v,\pi(v)} \;:\; \pi : V \to V \text{ a cyclic permutation} \right\}$$

This is the analog of the Travelling Salesman problem (TSP), but in a directed graph. For our purposes we will assume we are given a complete graph. We will further assume that the triangle inequality holds, that is $\forall x, y, z \in V \;\; w_{x,y} \leq w_{x,z} + w_{z,y}$.

1.1 Notation

For any $U \subseteq V$

$$\begin{aligned} \delta^+(U) &= \{\ (u, w) \ : \ u \in U, w \notin U\ \} \\ \delta^-(U) &= \{\ (w, u) \ : \ u \in U, w \notin U\ \} \end{aligned}$$

And $\forall v \in V$ we write $\delta^+(v) = \delta^+(\{v\})$ and $\delta^-(v) = \delta^-(\{v\})$. For any vector $x \in \mathbb{R}_+^A$, and $F \subseteq A$ we use the notation $x(F) = \sum_{a \in F} x_a$.

1.2 Held-Karp LP Relaxation

The Held-Karp linear programming relaxation of ATSP is:

$$\begin{aligned} \min\ & w^\mathsf{T} x \\ \text{s.t. }\ & x(\delta^-(v)) = x(\delta^+(v)) \quad \forall v \in V && \text{Eulerian} \\ & x(\delta^+(U)) \geq 1 \quad \forall U \text{ s.t. } \emptyset \neq U \subsetneq V && \text{Strongly Connected} \\ & 0 \leq x_a \leq 1 \quad \forall a \in A \end{aligned}$$

This LP has exponentially many constraints, in terms of the number of vertices, since the third constraint depends on each subset of the vertices, but we will show it can still be solved in polynomial time.

Claim 1.2. The Held-Karp LP Relaxation for ATSP can be solved in time polynomial in the number of vertices.

Proof. Recall that by searching the space in a binary fashion we can turn a problem of optimality into one of feasibility. By results discussed in previous lectures if we can find a poly-time separation oracle to determine whether a given x respects all of the constraints we can solve the problem in polynomial time. Note the second and last lines contain at most linearly many constraints, and can be checked in constant time, thus the bottle neck is the "Strongly Connected" constraints.
Note that the "Strongly Connected" constraints state "any cut of V has weight ≥ 1" which is equivalent to "the max s-t flow value ≥ 1 $\forall s, t \in V$" by the Max-Flow Min-Cut theorem. The flow interpretation of this constraint is a lot easier to verify, as there are only $O(n^2)$ pairs $s, t \in V$. Recall we came up with a linear program for solving this problem in an earlier lecture, and thus have a polynomial time algorithm for verifying a polynomial number of constraints.
We can use all of this to construct a separation oracle for the feasible region of this LP. First we check the constraints that can be checked in polynomial time, if any of them are not satisfied we return that constraint. Then we check if any pair $s, t \in V$ has max-flow less than 1. If it does than we can return the set U which has the minimum cut $\delta^+(U)$ for that s-t flow problem as the constraint that is violated.
Since we have a polynomial time separation oracle, this LP can be solved in polynomial time. $\square$

Next we discuss how we can interpret this LP as a statement about undirected graphs.

Claim 1.3. $\forall U \subseteq V$ $x(\delta^+(U)) = x(\delta^-(U))$

Proof. Note that from our constraints $\sum_{v \in U} x(\delta^+(v)) - x(\delta^-(v)) = 0$.
In order to transform this expression into the one in our claim, we examine the summation, by considering what contribution each individual edges has to the sum. There are four cases, in which we consider whether the endpoints of $v_1 v_2$ are in U or not.
Case 1: $v_1, v_2 \in U$. Then the contribution to the sum is 0, as $x_{v_1 v_2}$ is added as $x(\delta^+(v_1))$ and subtracted as $x(\delta^-(v_2))$.
Case 2: $v_1, v_2 \notin U$. Then the contribution to the sum is again 0, as $x_{v_1 v_2}$ is added in neither term.
Case 3: $v_1 \in U$ $v_2 \notin U$. In this case there is a $x_{v_1 v_2}$ term added as $x(\delta^+(v_1))$ but as $v_2 \notin U$ this is never subtracted away. Thus the contribution to the sum is $x_{v_1 v_2}$.
Case 4: $v_1 \notin U$ $v_2 \in U$. Similar to case 3, but the other way around, so the contribution to the sum is $-x_{v_1 v_2}$.

By this analysis we can see

$$\begin{aligned} 0 &= \sum_{v\in U} x(\delta^+(v)) - x(\delta^-(v)) \\ &= \sum_{a\in\delta^+(U)} x_a - \sum_{b\in\delta^-(U))} x_b \\ &= x(\delta^+(U)) - x(\delta^-(U)) \\ \implies \qquad x(\delta^+(U)) &= x(\delta^-(U)) \end{aligned}$$

□

Claim 1.4. $\exists$ undirected graph $G' = (V, E')$ with weights $y \in \mathbb{R}^{E'}$ such that $\forall U \subset V\ y(\delta(U)) = x(\delta^+(U))$. (Note that this is not true in general, but holds since we have a Eulerian graph)

Proof. Set $\forall v, w \in V\ y_{vw} = (x_{vw} + x_{wv})/2$. Then

$$y(\delta(U)) \ = \sum_{v\in U,\, w\notin U} \frac{x_{vw} + x_{vw}}{2} \ = \ \frac{1}{2}\Big(x(\delta^+(U)) + x(\delta^-(U))\Big) \ = \ x(\delta^+(U)),$$

where the last equality follows by Claim 1.3. □

Corollary 1.5. $\forall \alpha \geq 1\ |\{\ U\ :\ x(\delta^+(U)) \leq \alpha\ \}| < n^{2\alpha}$

Proof. This follows from our discussion of α min cuts, since the minimum cut of our graph G' is ≥ 1, by our strongly connected constraints. □

2 Rounding the LP Solution

When we get an integral solution to this LP we have a solution to the ATSP. However we may not get an integral solution, in which case we need to round our solution to get something that is integral. We next give an algorithm which does just that.

Let H be an empty digraph on V
Let $K := 200 \ln n$.
for all $i \in [K]$ **do**
  for all pairs of vertices $\in V$ **do**
    With probability x_{uv} add uv to H
  end for
end for

We will show that this gives an approximation which is $O(\log(n)\text{OPT})$, where OPT is the size of the optimal solution, with high probability.
This requires the use of the Markov inequality, which follows. For additional details, see Prof. Harvey's Lecture 2 Notes for Randomized Algorithms.

Theorem 2.1 (Markov's Inequality). Given a non-negative random variable X, and $\alpha \in \mathbb{R}^+$, $\Pr[X \geq \alpha E[X]] \leq \frac{1}{\alpha}$

Claim 2.2. With probability ≥ 0.9 $w(H) \leq O(\log(n)) w^\mathsf{T} x$

Proof. In order to use Markov's inequality we need to know the expectation of $w(H)$.

$$\begin{aligned}
E[w(H)] &= E[\sum_{uv \in A} \text{number of times } uv \text{ added to } H \times w_{uv}] \\
&= \sum_{uv \in A} E[\text{number of times } uv \text{ added to } H \times w_{uv}] \\
&= \sum_{uv \in A} w_{uv} E[\text{number of times } uv \text{ added to } H] \\
&= K \sum_{uv \in A} w_{uv} x_{uv} \\
&= K \cdot w^\mathsf{T} x
\end{aligned}$$

So $\Pr[w(H) > 10 K w^\mathsf{T} x] \leq \frac{1}{10}$ □

Next we begin a proof that the graph H is Strongly Connected with high probability.
That proof requires a knowledge of Chernoff bounds. For additional details, see Prof. Harvey's Lecture 2 Notes for Randomized Algorithms.

Theorem 2.3 (Chernoff Bound). Let $X_1, X_2, ..., X_n$ be independent random variables over the range $[0,1]$, $X = \sum_i X_i$, and $\mu = E[X]$. For any $\delta \in [0,1]$, then

$$\Pr[X \leq (1-\delta)\mu \text{ or } X \geq (1+\delta)\mu] \;\leq\; 2\exp\left(\frac{-\delta^2 \mu}{3}\right).$$

The Chernoff bound can be used to significantly decrease the failure probability in Claim 2.2, although we will not bother to do this. Next we show that the probability of any specific cut being more than 25% larger or smaller than its expectation is $O(\frac{1}{n^3})$.

Claim 2.4. Choose $U \subset V$, recall $\mu = E[|\delta^+(U)|]$, then $\Pr[|\delta^+(U)| \notin [\frac{3}{4}, \frac{5}{4}]\mu] \in O(\frac{1}{n^3})$

Proof. Choose $U \subset V$, let $Z_{a,i} = \begin{cases} 1 \text{ if arc } a \text{ is added in the } i^{th} \text{ iteration} \\ 0 \text{ otherwise} \end{cases}$

Note:

$$|\delta_H^+(U)| = \sum_{i=1}^{K} \sum_{a \in \delta^+(U)} Z_{a,i}$$

And so

$$\begin{aligned}
\mu &= E[|\delta_H^+(U)|] \\
&= \sum_{i=1}^{K} \sum_{a \in \delta^+(U)} E[Z_{a,i}] \\
&= \sum_{i=1}^{K} \sum_{a \in \delta^+(U)} x_a \\
&= K \cdot x(\delta_H^+(U))
\end{aligned}$$

Thus

$$\begin{aligned}
\Pr\left[|\delta_H^+(U)| \notin \left[\frac{3}{4}, \frac{5}{4}\right]\mu\right] &\leq 2\exp\big(-\mu/48\big) \\
&= 2\exp\big(-K \cdot x(\delta^+(U))/48\big) \\
&\leq 2\exp\big(-4\ln(n)x(\delta^+(U))\big) \\
&\leq n^{-3x(\delta^+(U))}
\end{aligned}$$

□

The previous claim shows that, for a *fixed* $U \subseteq V$, $|\delta^+(U)|$ is close to $E[|\delta^+(U)|]$ with high probability. In the next lecture we will show that, with high probability, for *every* $U \subseteq V$, $|\delta^+(U)|$ is close to $E[|\delta^+(U)|]$.