UBC CPSC 536N: Sparse Approximations	Winter 2013

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1 Combinatorial IPs

1.1 Mathematical programs

- Linear Program (LP): $\begin{cases} \min & c^\mathsf{T} x \\ \text{s.t.} & {a_i}^\mathsf{T} x \le b_i \ \ \forall i = 1, \cdots, m \end{cases}$

  LP can be efficiently solved by e.g Ellipsoid Method.

- Integer Program (IP) $\begin{cases} \min & c^\mathsf{T} x \\ \text{s.t.} & {a_i}^\mathsf{T} x \le b_i \ \ \forall i = 1, \cdots, m \\ x \in \mathbb{Z}^n & \end{cases}$

  IP cannot be efficiently solved assuming $P \neq NP$

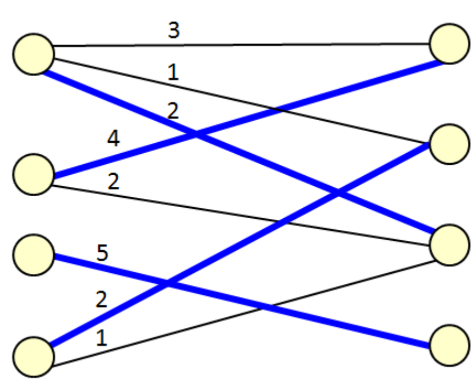
1.2 Combinatorial Optimization

Combinatorial optimization deals with study of optimization problems that have discrete solutions and some combinatorial flavor such as involving graphs. These problems have many applications in operation research, computer networks, compilers, online advertising and etc. Also in this area there exist rich theory of what can be solved efficiently and what cannot.

1.3 Example

Max weight perfect matching problem: Given bipartite graph $G = (V, E)$, every edge e has weight w_e. Find a maximum-weight perfect matching which is a set $M \subset E$ s.t. every vertex has exactly one incident edge in M.

Example 1.1. The blue edges are a max-weight perfect matching M.



The natural integer program:

$$\begin{cases} \max & \sum_{e \in E} w_e \cdot x_e \\ \text{s.t.} & \sum_{e \text{ incident to } v} x_e = 1 \quad \forall v \in V \\ & x_e \in \{0,1\} \qquad\qquad\quad \forall e \in E \end{cases}$$

This IP can be efficiently solved in many different ways.

1.4 How to solve combinatorial IPs?

Generally there are two common approaches for solving combinatorial IPs:

- Design combinatorial algorithm that directly solves IP. Often such algorithms have a nice LP interpretations.
- Relax IP to an LP; prove that they give same solution; solve LP by the ellipsoid method.
  - Need to show special structure of the LPs extreme points
  - Sometimes we can analyze the extreme points combinatorially
  - Sometimes we can use algebraic structure of the constraints. For example, if constraint matrix is Totally Unimodular then IP and LP are equivalent.

In this session we will see examples of these approaches.

2 Max Flow and Min Cut

2.1 Network Flow

Let $D = (N, A)$ be a directed graph. Every arc a has a capacity $c_a \geq 0$. We want to send oil from the node s to the node t through pipelines so that oil doesn't leak from any node except s and t which means that at these nodes *flow in = flow out*. For simplicity assume no arc enters s and no arc leaves t. The question is that how much oil can we send?

2.2 Max flow and Min Cut

Definition 2.1. Let $D = (N, A)$ be a digraph, where arc a has capacity c_a

For any $U \subseteq N$, the cut $\delta^+(U)$ is:

$$\delta^+(U) = \{ \ uv \ : \ u \in U, v \notin U, uv \in A \ \}$$

The capacity of the cut is:

$$c(\delta^+(U)) = \sum_{a \in \delta^+(U)} c_a$$

Lemma 2.2 (Weak Duality). For any flow and any U with $s \in U$, $t \notin U$, the amount of flow from s to t is at most $c(\delta^+(U))$

Proof. The net amount of flow crossing U is $\sum\limits_{a\in\delta^+(U)} x_a - \sum\limits_{a\in\delta^+(V\setminus U)} x_a$. Since $0 \le x_a \le c_a$:

$$\sum_{a\in\delta^+(U)} x_a - \sum_{a\in\delta^+(V\setminus U)} x_a \le \sum_{a\in\delta^+ U} c_a$$

□

We also have the strong duality theorem which is called the max-flow min cut-theorem:

Theorem 2.3 (Ford and Fulkerson 1956). The maximum amount of flow from s to t equals the minimum capacity of a cut $\delta^+(U)$ where $s \in U$ and $t \notin U$. Furthermore, if c is integral then there is an integral flow that achieves the maximum flow.

We are going to prove this theorem by linear programming.

2.3 LP formulation of Max Flow

- **Variables**: x_a is the amount of flow to send on arc a.
- **Constraints**: For every node except s and t, *flow in* = *flow out*
- **Objective value**: Total amount of flow sent by s.
- **Notation**: $\delta^+(v)$ = arcs with tail at v, $\delta^-(v)$ = arcs with head at v

The LP is:

$$\begin{cases} \max & \sum_{a\in\delta^+(s)} x_a & \\ \text{s.t} & \sum_{a\in\delta^-(v)} x_a - \sum_{a\in\delta^+(v)} x_a = 0 & \forall v \in V \setminus \{s,t\} \\ & 0 \le x_a \le c_a & \forall a \in A \end{cases}$$

The matrix M defining the constraints of this LP, such a matrix is called the incidence matrix of a directed graph, has one row for every node (except s or t) and one column for every arc. Note that every arc appears in two rows because its the tail of one vertex and the head of another vertex.

$$M_{v,a} = \begin{cases} +1 & \text{if node } v \text{ is the head of arc } a \\ -1 & \text{if node } v \text{ is the tail of arc } a \\ 0 & \text{otherwise} \end{cases}$$

The key property that this matrix has is that its a totally unimodular matrix.

3 Total Unimodularity

Definition 3.1. Let M be a real $m \times n$ matrix. Suppose that every square submatrix of M has determinant in $\{0, +1, -1\}$. Then M is totally unimodular.

In particular every entry of M must be in $\{0, +1, -1\}$ because every entry is a trivial square submatrix. Note that with this definition, polytopes defined by totally unimodular matrices have integral extreme points.

Lemma 3.2. Suppose M is totally unimodular matrix. Let b and c be integer vectors. Then

- Every extreme point of $P = \{\ x\ :\ Mx \leq b\ \}$ is integral.
- Every extreme point of $P = \{\ x\ :\ Mx = b, 0 \leq x \leq c\ \}$ is integral.

We know that extreme points are equivalent to basic feasible solutions so we prove lemma for basic feasible solutions:

Lemma 3.3. Suppose A is totally unimodular matrix. Let b be any integer vector. Then every basic feasible solution of $P = \{\ x\ :\ Ax \leq b\ \}$ is integral.

Proof. Let x be a basic feasible solution. Then the constraints that are tight at x have rank n. Let A' be a submatrix of A and b' a subvector of b corresponding to n linearly independent constraints that are tight at x.Then x is the unique solution to $A'x = b'$, i.e., $x = (A')^{-1}b$

Cramer's Rule: If M is a square, non-singular matrix then

$$(M^{-1})_{i,j} = (-1)^{i+j} \text{ det } M_{del(j,i)} / \text{ det } M$$

In which det $M_{del(j,i)}$ = submatrix of M obtained by deleting row j and column i.
Thus all entries of $(A')^{-1}$ are in $\{0, +1, -1\}$. Since b' is integral, x is also integral. □

Now we show that, as we claimed earlier, the matrix for the Max Flow problem is totally unimodular.

Lemma 3.4. Let $D = (N, A)$ be a directed graph. Define M by

$$M_{v,a} = \begin{cases} +1 & \text{if node } v \text{ is the head of arc } a \\ -1 & \text{if node } v \text{ is the tail of arc } a \\ 0 & \text{otherwise} \end{cases}$$

M is totally unimodular.

Proof. Let Q be a $k \times k$ submatrix of M. We argue by induction on k. If $k = 1$ then Q is a single entry of M, so $det\ Q$ is either 0 or ± 1. So assume $k > 1$.

Case 1: If some column of Q has no non-zero entries, then $det(Q) = 0$.

Case 2: Suppose j^{th} column of Q has exactly one non-zero entry, say $Q_{t,j} \neq 0$. Use column expansion of determinant:

$$det\ Q = \sum_i (-1)^{i+j} Q_{i,j} \cdot det\ Q_{del(i,j)} = (-1)^{t+j} Q_{t,j} \cdot det\ Q_{del(t,j)}$$

where t is the unique non-zero entry in column j. By induction, $det\ Q_{del(t,j)}$ in $\{0, +1, -1\} \Rightarrow$ $det\ Q$ in $\{0, +1, -1\}$

Case 3: Suppose every column of Q has exactly two non-zero entries. For each column one non-zero is a $+1$ and the other is a -1. So summing all rows in Q gives the vector $[0, 0, \cdots, 0]$. Thus Q is singular, and $det\ Q = 0$.

□

4 The Max Flow LP and its Dual

4.1 The Max Flow LP

The linear programming formulation:

$$\begin{cases} \max & \sum_{a\in\delta^+(s)} x_a \\ \text{s.t} & \sum_{a\in\delta^-(v)} x_a - \sum_{a\in\delta^+(v)} x_a = 0 \quad \forall v \in V \setminus \{s,t\} \\ & 0 \le x_a \le c_a \quad \forall a \in A \end{cases}$$

Assuming that the capacities are all non-negative, the LP is feasible since the zero flow is feasible. The feasible region is bounded thus the LP is bounded. So by fundamental theorem of linear programming it has an optimal solution which is a maximum flow. The feasible region is the polyhedron $P = \{\, x : Mx = b, 0 \le x \le c \,\}$ where M is totally unimodular matrix. Note that since P is bounded it doesn't contain any line and it has extreme points.

Corollary 4.1. If c is integral then every extreme point is integral, and so there is a maximum flow that is integral.

So this gives us the second half of max-flow min-cut theorem that there is always an optimal flow that is integral. To prove the rest of the theorem we consider the dual. To find the dual for the max flow LP we consider a variable y_v for every $v \in N \setminus \{s,t\}$ and a variable z_{uv} for every arc uv. The dual is:

$$\begin{cases} \min & \sum_{a\in A} c_a z_a \\ \text{s.t.} & -y_u + y_v + z_{uv} \ge 0 \quad \forall uv \in A, v, w \in N \setminus \{s,t\} \\ & y_v + z_{sv} \ge 1 \quad \forall sv \in A \\ & -y_u + z_{ut} \ge 0 \quad \forall ut \in A \\ & z \ge 0 \end{cases}$$

Let's set $y_s = 1$ and $y_t = 0$ then the dual would be simplified as:

$$\begin{cases} \min & \sum_{a\in A} c_a z_a \ge 0 \\ \text{s.t.} & -y_u + y_v + z_{uv} \ge 0 \quad \forall uv \in A \\ & z \ge 0 \end{cases}$$

Where y_s and y_t are not variables $y_s = 1$ and $y_t = 0$.

We will show that given an given an optimal solution (y,z), we can construct a cut $\delta^+(U)$ such that $c(\delta^+(U)) = \sum\limits_{a\in A} c_a z_a$. In other words the capacity of the cut $\delta^+(U)$ equals the optimal value of the dual LP. By strong LP duality, this equals the optimal value of the primal LP, which is the maximum flow value. By weak duality every cut has capacity at least the max flow value so this must be a minimum cut.

$$\begin{aligned} \text{Primal:} \quad & \max\{\, d^\mathsf{T} x : Mx = 0, 0 \le x \le c \,\} \\ \text{Dual:} \quad & \min\left\{ c^\mathsf{T} z : \begin{pmatrix} M^\mathsf{T} & I \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \ge 0, z \ge 0, y_s = 1, y_t = 0 \right\} \\ & = \begin{cases} \min & \sum_{a\in A} c_a z_a \ge 0 \\ \text{s.t.} & -y_u + y_v + z_{uv} \ge 0 \quad \forall uv \in A \\ & z \ge 0 \quad y_s = 1, y_t = 0 \end{cases} \end{aligned}$$

We claim that $\begin{pmatrix} M^\mathsf{T} & I \end{pmatrix}$ is also totally unimodular. Therefore any extreme point solution of Dual has y and z integral. Since we are minimizing we can assume that $z_{uv} = \max\{y_u - y_v, 0\}$ for all arcs uv. Define $U = \{ v : y_v \geq 1 \}$; then $s \in U$ and $t \notin U$. Also note that $z_{uv} \geq 1$ for all $uv \in \delta^+(U)$.

$$\text{Max Flow value} \overset{\text{Strong Duality}}{=} \sum_a z_a c_a \quad \geq \quad c(\delta^+(U)) \overset{\text{Weak Duality}}{\geq} \text{Max Flow value}$$

Therefore $\delta^+(U)$ is a cut separating s and t with capacity equal to max flow.