UBC CPSC 536N: Sparse Approximations Winter 2013

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1 Approximating cuts via pipage rounding

Let $G = (V, E)$ be a graph and $n = |V|$. Let P be a spanning tree polytope of G. Given any fractional point x in P, we want to find an integral point tree such that for every cut, the tree approximates x.

Conjecture 1.1 (Goddyn, late 1980s). For all $x \in P$, there exists a spanning tree T such that

$$\underbrace{\chi_T(\delta(U))}_{|T \cap \delta(U)|} \leq O(1) \cdot x(\delta(U)) \qquad \text{for all } U \subseteq V.$$

The conjecture, if true, would have many implications in graph theory and algorithms.In particular, it would imply that ATSP can be approximated within a constant factor.

We will prove a weaker version of this conjecture (which is the best known result).

Theorem 1.2. For all $x \in P$, there exists a spanning tree T such that

$$\chi_T(\delta(U)) \leq O(\frac{\log n}{\log \log n}) \cdot x(\delta(U)) \qquad \text{for all } U \subseteq V.$$

In order to prove this theorem, we run pipage rounding algorithm on x. This gives a tree, and we show that with good probability the inequality is satisfied. If we specify a partially concave function f that somehow relates to the cuts, then we can use f to give us the desired inequality. So we need to design f. Let x_1 be the starting x in the algorithm.

Definition 1.3. For $U \subseteq V$, define $P_U : P \to \mathbb{R}$ by

$$P_U(x) = e^{-\lambda\alpha} \cdot \prod_{e \in \delta(U)} (1 + x(e) \cdot e^{\lambda}),$$

where $\alpha = \frac{6 \log n}{\log \log n} \cdot x_1(\delta(U))$ and $\lambda = \log(\frac{\alpha}{x_1(\delta(U))})$.

Claim 1.4. P_U is partially concave.

Proof. Fix $x \in P$ and coordinates i and j. The univariate function $z \mapsto P_U(x + z(e_i - e_j))$ is continuous. So we just need to compute its second derivative and if that is non positive, then the univariate function is concave.

Note that if $i \notin \delta(U)$ or $j \notin \delta(U)$, then the second derivative would be zero. So we assume that $i, j \in \delta(U)$. Thus, $\frac{d^2}{dz^2} P_U(x + z(e_i - e_j))$ is

$$e^{-\lambda\alpha} \cdot [\prod_{\substack{e \in \delta(U) \\ e \notin \{i,j\}}} (1 + x(e) \cdot e^{\lambda})] \cdot \underbrace{\frac{d^2}{dz^2}[(1 + (x(i) + z)e^{\lambda})(1 + (x(j) - z)e^{\lambda})]}_{=-2e^{2\lambda}}.$$

Therefore $\frac{d^2}{dz^2} P_U(x + z(e_i - e_j))$ is non positive, which implies that P_U is partially concave. $\square$

Observation 1.5. $\lambda - 1 \geq \frac{\log\log n}{2}$, assuming $n > e$. (The logarithms are to base e.)

Proof. Note that $n > e$ is required in order to have λ well defined.

$$\begin{aligned}\lambda - 1 &= \log(\frac{6\log n}{\log\log n}) - 1 \\ &= \log 6 + \log\log n - \log\log\log n - 1\end{aligned}$$

We know that $e^x \geq x^2$ (and as a result $x \geq 2\log x$) for all positive x. For $n \geq e$, we have $\log\log n \geq 0$ and thus $\log\log n \geq 2\log\log\log n$. Therefore, $2\log 6 + \log\log n \geq 2 + 2\log\log\log n$, which concludes the observation. $\square$

Claim 1.6. $P_U(x_1) \leq n^{-3x_1(\delta(U))}$.

Proof.

$$\begin{aligned}
P_U(x_1) &= e^{-\lambda\alpha} \cdot \prod_{e\in\delta(U)} (1 + x_1(e) \cdot e^{\lambda}) \\
&\leq e^{-\lambda\alpha} \cdot \prod_{e\in\delta(U)} e^{x_1(e)\cdot e^{\lambda}} && [\text{by convexity inequality } 1 + x \leq e^x \quad \forall x \in \mathbb{R}] \\
&= \exp(-\lambda\alpha + e^{\lambda} \cdot \sum_{e\in\delta(U)} x_1(e)) \\
&= \exp(-\lambda\alpha + \alpha) = \exp(-\alpha(\lambda - 1)) && [e^{\lambda} = \frac{\alpha}{x_1(\delta(U))} \text{ and } \sum_{e\in\delta(U)} x_1(e) = x_1(\delta(U))] \\
&\leq \exp(-\frac{6\log n}{\log\log n} \cdot x_1(\delta(U)) \cdot \frac{\log\log n}{2}) && [\text{using Observation 1.5}] \\
&= \exp(-3\log n \cdot x_1(\delta(U))) \\
&= n^{-3x_1(\delta(U))}.
\end{aligned}$$

$\square$

So we chose P_U so that it is partially concave and proved an upper bound on its value at x_1. We know that pipage rounding gives us control over things by partially concave functions. However, P_U depends only on a single cut, but we need a function that controls all cuts simultaneously. So we define function f as follows.

Definition 1.7. $f(x) = \sum_{\emptyset\neq U\subset V} P_U(x)$

Note that f has exponentially many terms. However, this does not matter since the algorithm does not evaluate f.

Claim 1.8. $f(x_1) \leq \frac{3}{n}$.

Proof. In Lectures 8 and 9, we proved that if G is any undirected graph with weights x on the edges, and minimum cut at least one, then $\sum_{\emptyset\neq U\subset V} n^{-3x(\delta(U))} \leq \frac{3}{n}$.

So we just need to show that having weights x_1 on the edges, the minimum cut would be at least one. Note that x_1 is in the spanning tree polytope. We know that every tree in the graph is connected and hence has minimum cut at least one. Since x_1 is a convex combination of trees, we know $x_1 = \sum_{\text{all spanning trees } T_i} \alpha_{T_i} \cdot T_i$, where $\sum_{\text{all spanning trees } T_i} \alpha_{T_i} = 1$. As a result if inequality holds for trees, it also holds for x_1. This is because for all U we have

$$\begin{aligned}
\sum_{e\in\delta(U)} x_1(e) &= \sum_{e\in\delta(U)} \sum_{\substack{\text{all spanning trees}\\ T_i}} \alpha_{T_i} \cdot T_i(e) \\
&= \sum_{\substack{\text{all spanning trees}\\ T_i}} [\alpha_{T_i} \cdot \sum_{e\in\delta(U)} T_i(e)] \\
&\geq \sum_{\substack{\text{all spanning trees}\\ T_i}} \alpha_{T_i} \\
&= 1.
\end{aligned}$$

□

Claim 1.9. f is partially concave.

Proof. Since P_U and f are both continuous, we can look at the second derivatives, for fixed $x, i,$ and j.

$$\frac{d^2}{dz^2} f(x + z(e_i - e_j)) = \sum_{\emptyset \neq U \subset V} \frac{d^2}{dz^2} P_U(x + z(e_i - e_j)) \leq 0.$$

□

We now run the pipage rounding algorithm with input $x_1 = x$, and with this partially concave function f. Pipage rounding gives an extreme point $x^\star = \chi_T$ with $E[f(\chi_T)] \leq f(x_1) \leq \frac{3}{n}$.

Claim 1.10. $Pr[f(\chi_T) \leq 1] \geq 1 - \frac{3}{n}$.

Proof. By Markov's inequality, we get $Pr[f(\chi_T) \geq 1] \leq \frac{E[f(\chi_T)]}{1} \leq \frac{3}{n}$. □

We showed that, with good probability, f of final tree is small. Next, we use this to infer that the inequality in Theorem 1.2 holds for all cuts.

Claim 1.11. With probability $1 - \frac{3}{n}$, for all $U \subseteq V$ we have $\chi_T(\delta(U)) \leq O(\frac{\log n}{\log\log n}) \cdot x(\delta(U))$.

Proof. We know that with probability at least $1 - \frac{3}{n}$, $f(\chi_T) \leq 1$ (by Claim 1.10). Each P_U is non negative. Hence by definition of f, with probability at least $1 - \frac{3}{n}$, we have

$$1 \geq P_U(\chi_T) \;=\; e^{-\lambda\alpha} \cdot \prod_{e\in\delta(U)} (1 + \chi_T(e) \cdot e^{\lambda}) \;=\; e^{-\lambda\alpha} \cdot \prod_{e\in\delta(U)\cap T} (1 + e^{\lambda}).$$

Thus,

$$e^{\lambda\alpha} \geq \prod_{e\in\delta(U)\cap T} e^{\lambda} = \exp(|\delta(U) \cap T| \cdot \lambda),$$

which implies $\lambda\alpha \geq \lambda \cdot |\delta(U) \cap T|$. Therefore,

$$|\delta(U) \cap T| \leq \alpha = \frac{6 \log n}{\log \log n} \cdot x_1(\delta(U)).$$

This inequality holds for all $U \subseteq V$. $\square$

This concludes the proof of Theorem 1.2. The theorem implies a $O(\frac{\log n}{\log \log n})$ approximation for TSP problem, as we will discuss in Lecture 13.

2 Some remarks on pipage rounding algorithm

2.1 How do we get the maximal chain?

We can modify the algorithm to maintain the maximal chain during execution.

Initially the chain is the empty set and the entire set. We pick a direction and move in that direction. What prevents from moving further than z^+ or z^- is a new constraint. So when we move to $x + Z(e_a - e_b)$, we hit a new tight set. By argument of Lemma 2.7 in Lecture 10, we can add that new tight set to the chain and enlarge the chain.

2.2 How to compute z^+ and z^-?

We give two answers for this question.

Answer 1: Fix $x \in \mathbb{R}^E$. Assume $x \geq 0$ and $x(E) = n - 1$. Define $g(C) = r(C) - x(C)$. We know that $x \in P$ if and only if $x(C) \leq r(C)$ for all $C \subseteq E$. Thus,

$$x \in P \iff g(C) \geq 0 \;\; \forall C \subseteq E.$$

Remember that r is a submodular function, and x is a linear function. It is easy to show that g is also submodular. In other words, we have

$$x \in P \iff min\{g(C) : C \subseteq E\} \geq 0,$$

and deciding whether $x \in P$ is a submodular function minimization problem, which can be done in polynomial time. Then we can compute z^+ and z^- by binary search.

Answer 2: As with the ATSP LP, we can use minimum s-t cuts to give a separation oracle for P. Then again binary search gives us z^+ and z^-.