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Additional Notes for Lecture 4

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1 Covering Hemispheres by Ellipsoids

Recall our notation $B = \{ x : \|x\| \leq 1 \}$ and $H_u = \{ x : x^{\mathsf{T}} u \geq 0 \}$, where u is an arbitrary unit vector. The next theorem defines an ellipsoid that covers $B \cap H_u$ and analyzes its volume. For simplicity, let us assume that $n \geq 3$.

Theorem 1. Define

$$\begin{aligned} M &= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1} u u^{\mathsf{T}} \Big) \\ b &= \frac{u}{n+1} \\ B' &= E(M,b) = \Big\{ x : (x-b)^{\mathsf{T}} M^{-1} (x-b) \leq 1 \Big\}. \end{aligned}$$

Then B' satisfies the following two properties.

$$B \cap H_u \subseteq B' \tag{1}$$

$$\frac{\mathrm{vol}(B')}{\mathrm{vol}(B)} \leq e^{-1/4(n+1)} \leq 1 - \frac{1}{8(n+1)} \tag{2}$$

The following two claims prove the theorem.

Claim 2. $B \cap H_u \subseteq B'$.

Proof. First note that we can derive an explicit expression for M^{-1} using our Claim 1 on rank-1 updates.

$$M^{-1} = \frac{n^2-1}{n^2}\Big(I + \frac{2}{n-1} u u^{\mathsf{T}} \Big)$$

Substitute this into the definition of $E(M,b)$:

$$\begin{aligned} B' &= \left\{ x : \Big(x - \frac{u}{n+1} \Big)^{\mathsf{T}} \Big(\frac{n^2-1}{n^2} \Big) \Big(I + \frac{2}{n-1} u u^{\mathsf{T}} \Big) \Big(x - \frac{u}{n+1} \Big) \leq 1 \right\} \\ &= \left\{ x : \Big(x - \frac{u}{n+1} \Big)^{\mathsf{T}} \Big(x - \frac{u}{n+1} \Big) + \frac{2}{n-1} \left(u^{\mathsf{T}} \Big(x - \frac{u}{n+1} \Big) \right)^2 \leq \frac{n^2}{n^2-1} \right\} \\ &= \left\{ x : x^{\mathsf{T}} x - \frac{2x^{\mathsf{T}} u}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1} \Big(x^{\mathsf{T}} u - \frac{1}{n+1} \Big)^2 \leq 1 + \frac{1}{n^2-1} \right\} \end{aligned} \tag{3}$$

Now consider any $x \in B \cap H_u$. If we can show that $x \in B'$, then the proof is complete. By (3), it is sufficient to show that

$$x^{\mathsf{T}} x - \frac{2x^{\mathsf{T}} u}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1} \Big(x^{\mathsf{T}} u - \frac{1}{n+1} \Big)^2 \leq 1 + \frac{1}{n^2-1}.$$

We know that $x^\mathsf{T} x \leq 1$ (since $x \in B$), so it is sufficient to show that

$$\underbrace{-\frac{2x^\mathsf{T} u}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1}\Big(x^\mathsf{T} u - \frac{1}{n+1}\Big)^2}_{f(x^\mathsf{T} u)} \;\leq\; \frac{1}{n^2-1}. \tag{4}$$

The left-hand side of (4) is a function only of $x^\mathsf{T} u$, so let's call it $f(x^\mathsf{T} u)$. Every point in $B \cap H_u$ satisifies $0 \leq x^\mathsf{T} u \leq 1$, by the Cauchy-Schwarz inequality. For notational simplicity, let y denote the scalar $x^\mathsf{T} u$. So, to prove (4), we must analyze the maximum value of $f(y)$ on the interval $[0, 1]$. Note that f is a quadratic polynomial in y and it is convex (i.e., the coefficient of y^2 is positive), so f is maximized on $[0, 1]$ either at $y = 0$ or $y = 1$. We compute

$$f(0) \;=\; \frac{1}{(n+1)^2} + \frac{2}{n-1}\cdot\frac{1}{(n+1)^2} \;=\; \frac{1}{(n+1)^2}\Big(1 + \frac{2}{n-1}\Big) \;=\; \frac{1}{(n+1)^2}\cdot\frac{n+1}{n-1} \;=\; \frac{1}{n^2-1}.$$

On the other hand,

$$\begin{aligned}
f(1) &= \frac{-2}{n+1} + \frac{1}{(n+1)^2} + \frac{2}{n-1}\Big(\frac{n}{n+1}\Big)^2 \\
&= \frac{1}{(n+1)^2(n-1)}\Big(-2(n+1)(n-1) + n - 1 + 2n^2\Big) \\
&= \frac{1}{(n+1)^2(n-1)}\Big(-2(n+1)(n-1) + (n+1)(2n-1)\Big) \\
&= \frac{1}{n^2-1}.
\end{aligned}$$

This proves (4), and so $B \cap H_u \subseteq B'$. ☐

Claim 3. $\operatorname{vol}(B') \leq \operatorname{vol}(B) \cdot e^{-1/4(n+1)}$.

Proof. Note that $B' = E(M, b) = f(B)$ where f is the affine map $f(x) = M^{1/2}x + b$. The map f scales volumes by $\sqrt{|\det M|}$, so

$$\begin{aligned}
\left(\frac{\operatorname{vol}(B')}{\operatorname{vol}(B)}\right)^2 &= |\det M| \\
&= \left(\frac{n^2}{n^2-1}\right)^n\Big(1 - \frac{2}{n+1}\Big) && \text{(from facts on rank-1 updates)} \\
&= \left(1 + \frac{1}{n^2-1}\right)^n\Big(1 - \frac{2}{n+1}\Big) \\
&\leq \left(\exp\Big(\frac{1}{n^2-1}\Big)\right)^n \cdot \exp\Big(-\frac{2}{n+1}\Big) && \text{(since } 1 + x \leq e^x \text{ for all } x\text{)} \\
&= \exp\left(\frac{n}{(n+1)(n-1)} - \frac{2}{n+1}\right) \\
&= \exp\left(\frac{1}{n+1}\Big(\frac{n}{n-1} - 2\Big)\right) \\
&\leq \exp\left(\frac{-1}{2(n+1)}\right) && \text{(since } n \geq 3\text{)}
\end{aligned}$$

Taking square roots proves the claim. ☐

2 Covering Half-ellipsoids by Ellipsoids

Let $E = E(N, z)$ be an ellipsoid, where N is a positive definite matrix and z is a vector. Let $H_a = \{\, x : a^\mathsf{T}(x - z) \geq 0 \,\}$ be a halfspace with z on its boundary. We would like to find a small ellipsoid E' containing the half-ellipsoid $E \cap H_a$. To solve this problem, we will use our previous result on covering hemispheres by ellipsoids. We would like to find a linear map f and choose u such that:

- $f(B) = E$
- $f(H_u) = H_a$. (Recall that $H_u = \{\, x : u^\mathsf{T} x \geq 0 \,\}$.)

In Lecture 6, we showed that E can be obtained by applying the affine map $f(x) = N^{1/2}x + z$ to the unit ball B. Now if we can judiciously choose a unit vector u such that $f(H_u) = H_a$, then we'll be done. This turns out to be straightforward. Notice that

$$\begin{aligned} f(H_u) &= \left\{ N^{1/2}x + z \,:\, u^\mathsf{T} x \geq 0 \right\} \\ &= \left\{ x \,:\, u^\mathsf{T} N^{-1/2}(x - z) \geq 0 \right\}. \end{aligned}$$

So choose $u = N^{1/2}a$. Let B' to be the ellipsoid covering $B \cap H_u$, as given in Theorem 1. Our solution is to define $E' = f(B')$.

Claim 4. E' is an ellipsoid.

Proof. B' is an ellipsoid, so it is obtained by applying an affine map to the unit ball. E' is obtained by applying an affine map to B'. Composing these maps shows that E' can also be obtained by applying an affine map to the unit ball, so E' is also an ellipsoid. ■

Claim 5. $E \cap H_a \subseteq E'$.

Proof. Exercise. ■

Claim 6. $\mathrm{vol}(E') \leq \mathrm{vol}(E) \cdot e^{-1/4(n+1)}$.

Proof. Since $E' = f(B')$ and $E = f(B)$, our result on volumes under affine maps from Lecture 6 implies that

$$\begin{aligned} \mathrm{vol}(E') &= |\det(N^{1/2})|\,\mathrm{vol}(B') \\ \mathrm{vol}(E) &= |\det(N^{1/2})|\,\mathrm{vol}(B). \end{aligned}$$

Claim 3 above shows that $\mathrm{vol}(B') \leq \mathrm{vol}(B) \cdot e^{-1/4(n+1)}$. This proves the claim. ■

Rank-1 Updates

- **Def:** Let z be a column vector and α a scalar.
  A matrix of the form $I + \alpha z z^{\mathsf{T}}$ is called a **rank-1 update matrix**.

- **Claim 1:** Suppose $\alpha \neq -1/z^{\mathsf{T}}z$. Then
  $(I + \alpha z z^{\mathsf{T}})^{-1} = I + \beta z z^{\mathsf{T}}$ where $\beta = -\alpha/(1+\alpha z^{\mathsf{T}}z)$.

- **Claim 2:** If $\alpha \geq -1/z^{\mathsf{T}}z$ then $I + \alpha z z^{\mathsf{T}}$ is PSD.
  If $\alpha > -1/z^{\mathsf{T}}z$ then $I + \alpha z z^{\mathsf{T}}$ is PD.

- **Claim 3:** $\det(I + \alpha z z^{\mathsf{T}}) = 1 + \alpha z^{\mathsf{T}} z$