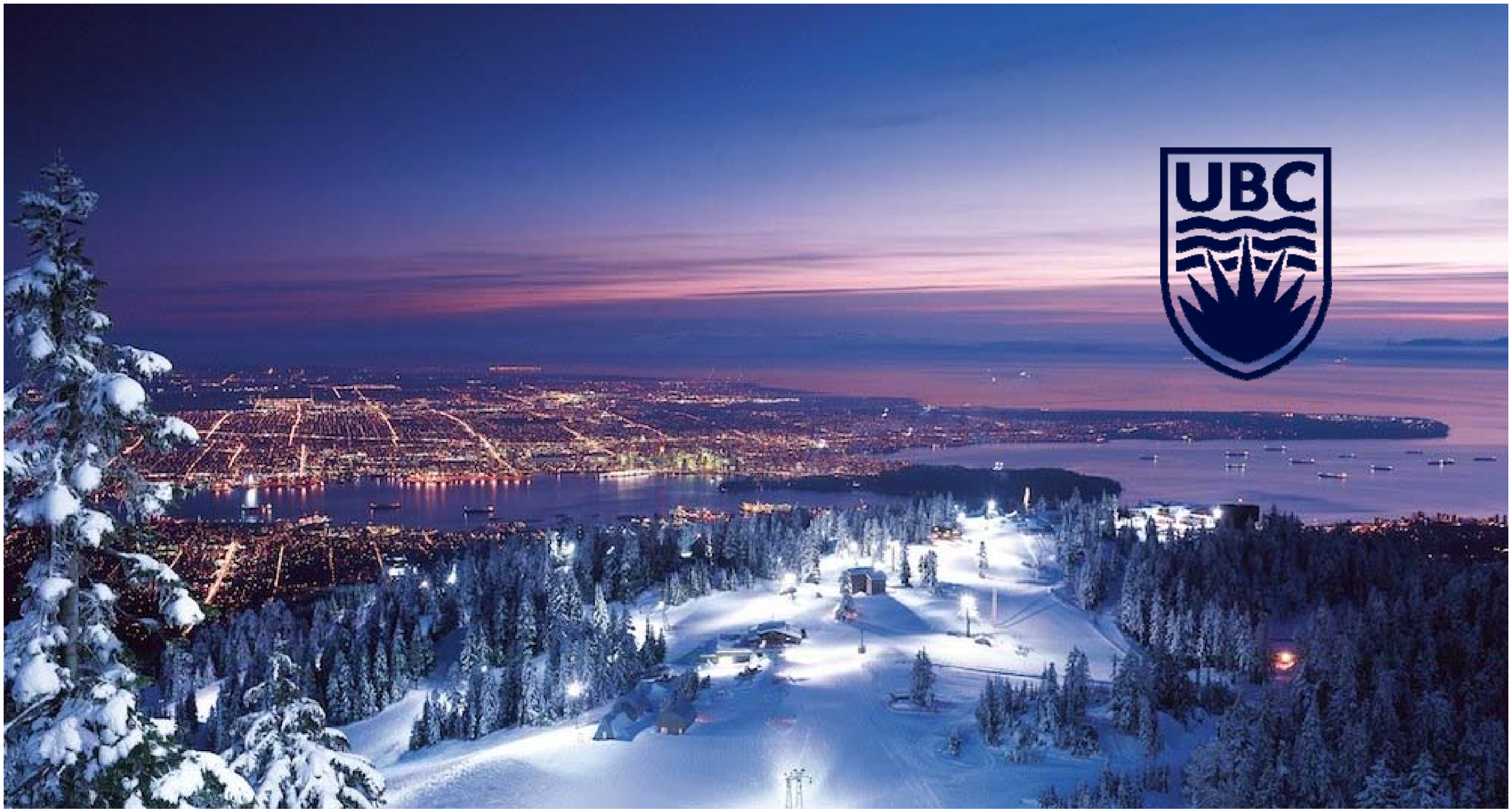
Approximating Submodular Functions



Nick Harvey
University of British Columbia

Approximating Submodular Functions
Part 1

Nick Harvey

University of British Columbia
Department of Computer Science

July 11th, 2015

Joint work with M. Goemans, S. Iwata and V. Mirrokni

Submodular Functions

- Studied for decades in combinatorial optimization and economics
- Used in approximation algorithms, algorithmic game theory, machine learning, etc.
- Discrete analogue of convex functions
  [Lovász '83], [Murota '03]

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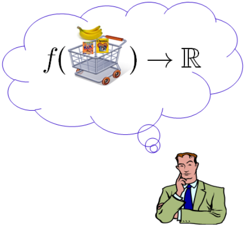
SUBMODULAR FUNCTIONS, MATROIDS, AND CERTAIN POLYHEDRA*

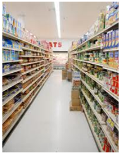
Jack Edmonds

National Bureau of Standards, Washington, D.C.,U.S.A.

Valuation Functions

- A first step in economic modeling: individuals have valuation functions giving utility for different outcomes or events





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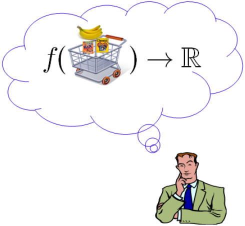

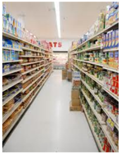

- Focus on combinatorial settings:
  - n items, $\{1, 2, ..., n\} = [n]$
  - $f : 2^{[n]} \to \mathbb{R}$
  - Submodularity is often a natural assumption

Approximating submodular functions based on few values

Motivating Example

- Microsoft Office consists of a set A of products.
  e.g., $A = \{\text{Word}, \text{Excel}, \text{Outlook}, \text{PowerPoint}, ...\}$.
- Consumer has a valuation function $f : 2^A \rightarrow \mathbb{R}$
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Outline

- Today: Actively querying the function
- Tomorrow: Observing samples from a distribution

Submodular Functions

Definition
$f : 2^{[n]} \to \mathbb{R}$ is submodular if, for all $A, B \subseteq [n]$:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

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Equivalent Definition

f is submodular if, for all $A \subseteq B$ and $i \notin B$:

$$f(A \cup \{i\}) - f(A) \geq f(B \cup \{i\}) - f(B)$$

Diminishing returns: the more you have, the less you want

Example

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Combinatorial Examples

- Linear algebra: Let $v_1, \ldots, v_n \in \mathbb{R}^d$. For $S \subseteq [n]$, let

$$f(S) = \dim \operatorname{span} \{ \, v_i \, : \, i \in S \, \} \, .$$

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- Matroids: Let $M = (E, \mathcal{I})$ be a matroid. For $S \subseteq E$, the rank function is

$$f(S) = \max \{\, |I| \,:\, I \subseteq S,\, I \in \mathcal{I} \,\}.$$

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- Coverage: Let $A_1, \ldots, A_n \subseteq U$. For $S \subseteq [n]$, define

$$f(S) = \max \left| \bigcup_{i \in S} A_i \right| .$$

Minimizing Submodular Functions
(Given Oracle Access)

- Can solve $\min_S f(S)$ in polynomial time (and oracle calls).
  First shown using the ellipsoid method.
  [Grotschel, Lovasz, Schrijver '81]
- Combinatorial, strongly-polynomial time algorithms known.
  [Schrijver '01], [Iwata, Fleischer, Fujishige '01], ...

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- Combinatorial, strongly-polynomial time algorithms known.
  [Schrijver '01], [Iwata, Fleischer, Fujishige '01], ...
- One of the most powerful algorithms in combinatorial optimization. Generalizes bipartite matching, matroid intersection, polymatroid intersection, submodular flow, ...
- Example: Matroid Intersection [Edmonds '70]
  Given matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$

$$\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(S) + r_2(E \setminus S) : S \subseteq E\}$$

Maximizing Submodular Functions
(Given Oracle Access)

Maximization

- Can approximate $\max_S f(S)$ to within $1/4$, assuming $f \geq 0$:
  Just pick S uniformly at random!
  [Feige, Mirrokni, Vondrák '07]

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  Just pick S uniformly at random!
  [Feige, Mirrokni, Vondrák '07]
- Can approximate $\max_S f(S)$ to within $1/2$, assuming $f \geq 0$.
  [Buchbinder, Feldman, Naor, Schwartz '12]

Monotone Functions

Definition

$f : 2^{[n]} \to \mathbb{R}$ is monotone if, for all $A \subseteq B \subseteq [n]$:

$$f(A) \leq f(B)$$

Constrained Maximization

- Let $M = (E, \mathcal{I})$ be a matroid.
  Assume $f : 2^E \to \mathbb{R}_{\geq 0}$ is monotone and submodular.
  Can approximate $\max_{S \in \mathcal{I}} f(S)$ to within $1 - 1/e$.
  [Calinescu, Chekuri, Pal, Vondrák '09], [Filmus, Ward '12]

Monotone Functions

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Problem

Given a monotone, submodular f, construct using $\text{poly}(n)$ *oracle queries a function $\hat{f}$ such that:*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S) \qquad \forall\, S \subseteq [n]$$

Approximation Quality

- How small can we make $\alpha(n)$?
- $\alpha(n) = n$ is trivial

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Our Positive Result

A deterministic algorithm that constructs $\hat{f}(S)$ with

- $\alpha(n) = \sqrt{n+1}$ for matroid rank functions f, or
- $\alpha(n) = O(\sqrt{n}\log n)$ for general monotone submodular f

Also, $\hat{f}$ is submodular: $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ for some scalars c_i.

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Our Negative Result

With polynomially many oracle calls, $\alpha(n) = \Omega(\sqrt{n}/\log n)$ (even for randomized algs)

Polymatroid

Definition

Given submodular f, polymatroid

$$P_f = \left\{ x \in \mathbb{R}^n_+ : \sum_{i \in S} x_i \leq f(S) \text{ for all } S \subseteq [n] \right\}$$

A few properties [Edmonds '70]:

- Can optimize over P_f with greedy algorithm
- Separation problem for P_f is submodular fctn minimization
- For monotone f, can reconstruct f:

$$f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$$

Our Approach: Geometric Relaxation

We know:

$$f(S) = \max_{x \in P_f} \langle 1_S, x \rangle$$

Suppose that:

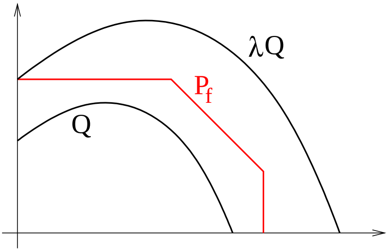
$$Q \subseteq P_f \subseteq \lambda Q$$

Then:

$$\hat{f}(S) \leq f(S) \leq \lambda \hat{f}(S)$$

where

$$\hat{f}(S) = \max_{x \in Q} \langle 1_S, x \rangle$$



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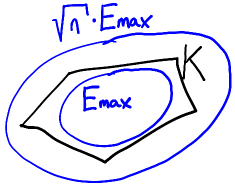
Theorem

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Let E_{max} (or John ellipsoid) be maximum volume ellipsoid contained in K. Then $K \subseteq \sqrt{n} \cdot E_{max}$.



John's Theorem [1948]
Maximum Volume Ellipsoids

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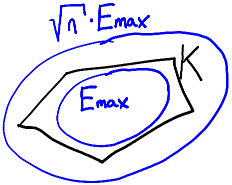


Algorithmically?

Ellipsoids Basics

Definition

- An ellipsoid is

$$E(A) = \{x \in \mathbb{R}^n : x^T A x \leq 1\}$$

where $A \succ 0$ is positive definite matrix.

Handy notation

- Write $\|x\|_A = \sqrt{x^T A x}$. Then

$$E(A) = \{x \in \mathbb{R}^n : \|x\|_A \leq 1\}$$

Optimizing over ellipsoids

- $\max_{x \in E(A)} \langle c, x \rangle = \|c\|_{A^{-1}}$

Algorithms for Ellipsoidal Approximations

Explicitly Given Polytopes

- Can find E_{max} in P-time (up to ϵ) if explicitly given as $K = \{x : Ax \leq b\}$
[Grötschel, Lovász and Schrijver '88], [Nesterov, Nemirovski '89], [Khachiyan, Todd '93], ...

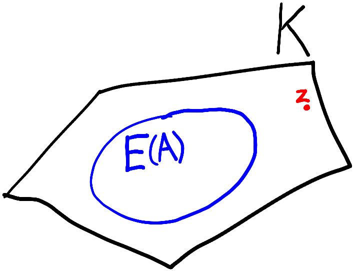
Polytopes given by Separation Oracle

- only $n+1$-ellipsoidal approximation for convex bodies given by weak separation oracle [Grötschel, Lovász and Schrijver '88]
- No (randomized) $n^{1-\epsilon}$-ellipsoidal approximation [J. Soto '08]

Finding Larger and Larger Inscribed Ellipsoids

Informal Statement

- We have $A \succ 0$ such that $E(A) \subseteq K$.
- Suppose we find $z \in K$ but z far outside of $E(A)$.
- Then should be able to find $A' \succ 0$ such that
  - $E(A') \subseteq K$
  - $\operatorname{vol} E(A') > \operatorname{vol} E(A)$



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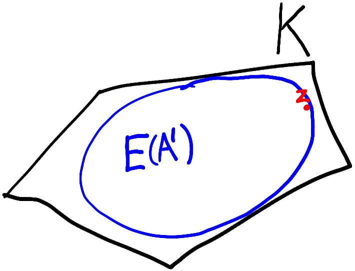


Finding Larger and Larger Inscribed Ellipsoids

Formal Statement

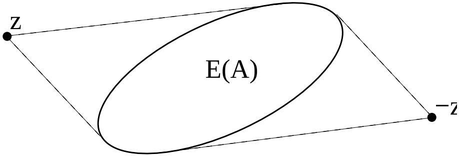
Theorem

If $A \succ 0$ *and* $z \in \mathbb{R}^n$ *with* $d = \|z\|_A^2 \geq n$ *then* $E(A')$ *is max volume ellipsoid inscribed in* $\mathrm{conv}\{E(A), z, -z\}$ *where*

$$A' = \frac{n}{d}\frac{d-1}{n-1}A + \frac{n}{d^2}\left(1 - \frac{d-1}{n-1}\right) Azz^T A$$

Moreover, $\mathrm{vol}\, E(A') = k_n(d) \cdot \mathrm{vol}\, E(A)$ *where*

$$k_n(d) = \sqrt{\left(\frac{d}{n}\right)^n \left(\frac{n-1}{d-1}\right)^{n-1}}$$



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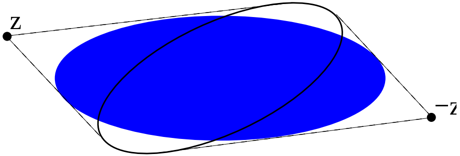

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Remarks

- $k_n(d) > 1$ for $d > n$ proves John's theorem
- Significant volume increase for $d \geq n+1$:
  $k_n(n+1) = 1 + \Theta(1/n^2)$
- Polar statement previously known [Todd '82]
  A' gives formula for minimum volume ellipsoid containing
  $$E(A) \cap \{\, x \,:\, -b \leq \langle c, x \rangle \leq b \,\}$$

Review of Plan

- Given monotone, submodular f, make $n^{O(1)}$ queries, construct $\hat{f}$ s.t.

$$\hat{f}(S) \leq f(S) \leq \tilde{O}(\sqrt{n}) \cdot \hat{f}(S) \qquad \forall\, S \subseteq V.$$

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- Compute ellipsoids $E_1, E_2, \ldots$ in $S(P_f)$ that converge to E_{max}.
Given $E_i = E(A_i)$, need $z \in S(P_f)$ with $\|z\|_{A_i} \geq \sqrt{n+1}$.
    - If $\exists z$, can compute E_{i+1} of larger volume.
    - If $\nexists z$, then $E_i \approx E_{max}$.

- Ellipsoidal Norm Maximization

  Given $A \succ 0$ and well-bounded convex body K by separation oracle.
  (So $B(r) \subseteq K \subseteq B(R)$ where $B(d)$ is ball of radius d.)
  Solve

$$\max_{x \in K} \|x\|_A$$

- **Ellipsoidal Norm Maximization**
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- **Bad News**
  Ellipsoidal Norm Maximization NP-complete for $S(P_f)$ and P_f.
  (Even if f is a graphic matroid rank function.)

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  (Even if f is a graphic matroid rank function.)

- Approximations are good enough

  P-time α-approx. algorithm for Ellipsoidal Norm Maximization
  $\implies$ P-time $\alpha\sqrt{n+1}$-ellipsoidal approximation for K
  (in $O(n^3 \log(R/r))$ iterations)

Our Task

Given $A \succ 0$, and f find $\max_{x \in S(P_f)} \|x\|_A$.

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Observation: Symmetry Helps

$S(P_f)$ invariant under axis-aligned reflections.

(Diagonal $\{\pm 1\}$ matrices.)

$\implies$ same is true for E_{max}

$\implies$ $E_{max} = E(D)$ where D is diagonal.

Our Task

Given diagonal $D \succ 0$, and f find

$$\max_{x \in S(P_f)} \|x\|_D$$

Equivalently,

$$\begin{array}{ll} \max & \sum_i d_i x_i^2 \\ \text{s.t.} & x \in P_f \end{array}$$

- Maximizing convex function over convex set
  $\Rightarrow$ max attained at vertex.

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Matroid Case

If f is matroid rank function
$\implies$ vertices in $\{0,1\}^n \implies x_i^2 = x_i$.
Our task is

$$\begin{array}{ll} \max & \sum_i d_i x_i \\ \text{s.t.} & x \in P_f \end{array}$$

This is the max weight base problem, solvable by greedy algorithm.

Our Task

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General Monotone Submodular Case

More complicated: uses approximate maximization of submodular function [Nemhauser, Wolsey, Fischer '78], etc.
Can find $O(\log n)$-approximate maximum.

Summary of Algorithm

Theorem

In P-time, construct a (submodular) function $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ *with*

- $\alpha(n) = \sqrt{n+1}$ *for matroid rank functions* f*, or*
- $\alpha(n) = O(\sqrt{n} \log n)$ *for general monotone submodular* f*.*

The algorithm is deterministic.

$\Omega(\sqrt{n}/\log n)$ Lower Bound

Theorem

With poly(n) *queries, cannot approximate f better than* $\frac{\sqrt{n}}{\log n}$*.*
Even for randomized algs, and even if f is matroid rank function.

$\Omega(\sqrt{n}/\log n)$ Lower Bound
Informal Idea

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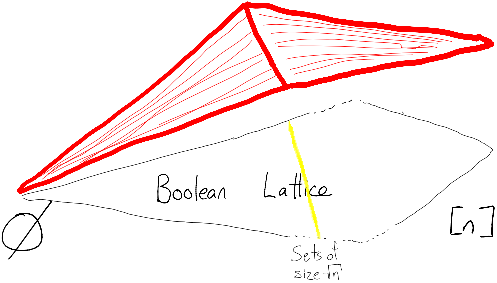

$f(S) = \min\{|S|, \sqrt{n}\}$

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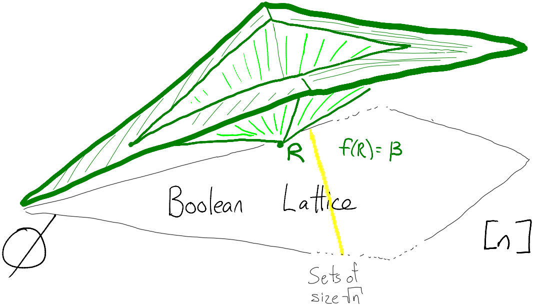
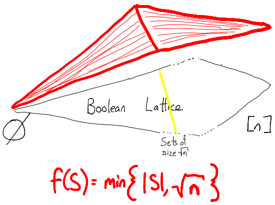
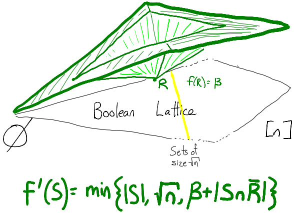


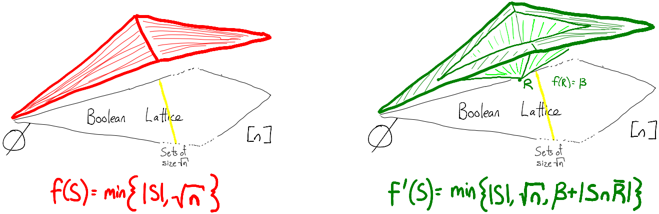
$$f'(S) = \min\{|S|, \sqrt{n}, \beta + |S \cap \bar{R}|\}$$





Algorithm performs queries $S_1, \ldots, S_k$. It distinguishes f from f' iff for some i,

$$\beta + |S_i \cap \bar{R}| < \min\{|S_i|, \sqrt{n}\}. \tag{1}$$



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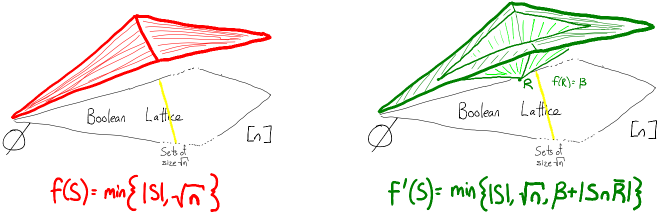

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Suppose $S_i \leq \sqrt{n}$. Then (1) holds iff $|S_i \cap R| > \beta$.
Pick R at random, each element w.p. $1/\sqrt{n}$.

$$\mathrm{E}\left[\,|S_i \cap R|\,\right] \;=\; |S_i|/\sqrt{n} \;\leq\; 1$$

$$\text{Chernoff bound} \implies \Pr\left[\,|S_i \cap R| > \beta\,\right] \;\leq e^{-\beta/2} = n^{-\Theta(1)}$$

Union bound implies that no query distinguishes f from f'.

Summary

Problem

Given a monotone, submodular f, construct using poly(n) *oracle queries a function $\hat{f}$ such that:*

$$\hat{f}(S) \leq f(S) \leq \alpha(n) \cdot \hat{f}(S) \qquad \forall\, S \subseteq [n]$$

Our Positive Result

A deterministic algorithm that constructs $\hat{f}(S) = \sqrt{\sum_{i \in S} c_i}$ with

- $\alpha(n) = \sqrt{n+1}$ for matroid rank functions f, or
- $\alpha(n) = O(\sqrt{n} \log n)$ for general monotone submodular f

Our Negative Result

With polynomially many oracle calls, $\alpha(n) = \Omega(\sqrt{n} / \log n)$
(even for randomized algs)

Extensions

Existential Approximations

Consider the statement: for a non-negative function $f : 2^{[n]} \rightarrow \mathbb{R}$ with $f(\emptyset) = 0$, there exists a "simple" function $\hat{f}$ with $\hat{f}(S) \leq f(S) \leq \sqrt{n} \cdot \hat{f}(S)$.

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Open Question

True for any submodular function?

Other Notions of Approximation

Suppose $f : 2^{[n]} \to [0,1]$ is submodular.
Then f is within ℓ_2-distance ϵ to

- a function of $O(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon})$ variables (a "junta"). [Feldman-Vondrak '13]
- a polynomial of degree $O(\frac{1}{\epsilon^2})$. [Cheragchi et al. '11]
- a polynomial of degree $O(\frac{1}{\epsilon^{4/5}} \log \frac{1}{\epsilon})$. Moreover, the "4/5" is optimal! [Feldman-Vondrak '15]