CPSC 421: Introduction to Theory of Computing
Assignment #8, due Wednesday November 23rd by 11:59pm, via Gradescope

[5] 1. Let $X = Y = \{0,1\}^3$. Here is the table for the equality function $EQ_3 : X \times Y \to \{0,1\}$.

	000	001	010	011	100	101	110	111
000	1	0	0	0	0	0	0	0
001	0	1	0	0	0	0	0	0
010	0	0	1	0	0	0	0	0
011	0	0	0	1	0	0	0	0
100	0	0	0	0	1	0	0	0
101	0	0	0	0	0	1	0	0
110	0	0	0	0	0	0	1	0
111	0	0	0	0	0	0	0	1

Find a *partition* of $X \times Y$ into 16 monochromatic rectangles. (The rectangles must be disjoint, and must cover $X \times Y$.) Express these rectangles in mathematical notation, not by drawing them.

[10] 2. Define

$$\begin{aligned} ODD &= \{ x \in \{0,1\}^n \,:\, x \text{ has an odd number of ones} \} \\ EVEN &= \{ x \in \{0,1\}^n \,:\, x \text{ has an even number of ones} \} \end{aligned}$$

Alice is given a vector $x \in ODD$ and Bob is given a vector $y \in EVEN$. Their goal is to output *any single* index $i \in \{1,\ldots,n\}$ for which $x_i \neq y_i$. (Such a coordinate must exist since obviously $x \neq y$.) Design a communication protocol for this that uses $O(\log n)$ bits of communication.

[10] 3. Let n, k be integers with $1 \leq k \leq n/2$. Let

$$\mathcal{X} \;=\; \{ X \,:\, X \subseteq \{1,\ldots,n\} \text{ and } |X| = k \} \;=\; \mathcal{Y}.$$

Let $f : X \times Y \to \{0,1\}$ be the disjointness function. That is, for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, let

$$f(X,Y) \;=\; \begin{cases} 1 & (\text{if } X \cap Y = \emptyset) \\ 0 & (\text{otherwise}) \end{cases}$$

Prove that the deterministic communication complexity $D(f)$ is at least $\lceil \log(n/k) \rceil$.

[10] 4. Recall the "combinatorial auction" problem. The items for auction are $U = \{1,\ldots,n\}$. There are two bidders, and bidder i has valuation function $v_i : 2^U \to \mathbb{R}$. (For simplicity, let us even assume $v_i : 2^U \to \{0,1\}$.) The goal is to compute $\max_{S \subseteq U}(v_1(S) + v_2(U \setminus S))$.

[10] a. In this question, we will prove that the bidders must exchange at least 2^n bits in order to solve the combinatorial auction problem. To do so, perform a reduction from the Disjointness problem. (Specifically, consider $DISJ_\ell$ with $\ell = 2^n$. Alice receives $A \subseteq \{1,\ldots,\ell\}$ and Bob receives $B \subseteq \{1,\ldots,\ell\}$. They must decide whether $A \cap B = \emptyset$.)
Hint: Let $\pi : 2^U \to \{1,\ldots,\ell\}$ be an arbitrary bijection. Define a valuation function for Alice using A and π, and define a valuation function for Bob using B and π.

[2] b. **OPTIONAL BONUS QUESTION**

It is a natural assumption in auctions that valuation functions are **monotone** (i.e., $v_i(A) \le v_i(B)$ if $A \subseteq B$). Intuitively, one should place more value on owning more items. Prove that a $2^{\Omega(n)}$ lower bound holds, even under the restriction that v_1 and v_2 are both monotone.

Hint: Let $\ell = \binom{n}{n/2}$. Consider only subsets A and B with $|A| = |B| = n/2$.

[3] 5. **OPTIONAL BONUS QUESTION:** Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be given by $f(x,y) = 1$ iff $\sum_i x_i y_i \equiv 0 \pmod{2017}$. Prove that f has no fooling set larger than n^c, for some constant c.

Relevant ideas are contained in Sherstov's lecture 3.
http://www.cs.ucla.edu/~sherstov/teaching/2012-winter/docs/lecture03.pdf.