CPSC 421: Introduction to Theory of Computing
Assignment #0, due Monday September 12th, in class

[5] 1. Let L_1 and L_2 be finite sets. Suppose that C is a finite set with $L_1 \in C$ and $L_2 \in C$. Suppose that $x \in L_1 \cap L_2$. Is $x \in C$? If so, explain why; if not, give a counterexample.

[10] 2. Let A and B be finite sets. A *bijection* from A to B is a function $f : A \to B$ that is:

- **Injective (or "one-to-one"):** if $f(a) = f(a')$ then $a = a'$,
- **Surjective (or "onto"):** for every $b \in B$ there exists $a \in A$ with $f(a) = b$.

[5] a. Suppose that $|A| = |B| = n \geq 1$. How many bijections from A to B are there?

[5] b. Suppose that $|A| = n$ and $|B| = 2n$ for $n \geq 1$. How many bijections from A to B are there?

[5] 3. Let $f(x)$, $g(x)$ and $h(x)$ be univariate polynomials of degree d. What is the degree of $f(g(h(x)))$?

[5] 4. Suppose we flip a biased coin that comes up heads with probability p and tails with probability $1 - p$. Suppose we perform k independent flips of this coin. What is the probability that we see heads at least once during these k flips?

[5] 5. Consider the following algorithmic problem. The input is a graph with n nodes (and no multi-edges). A *rectangle* in the graph is a sequence of four vertices (v_1, v_2, v_3, v_4) such that the edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_1, v_4\}$ are all present in the graph. (We do not care whether $\{v_1, v_3\}$ or $\{v_2, v_4\}$ are edges.)

Give an algorithm to decide if the given graph contains a rectangle. The running time of your algorithm should be polynomial in n. You may assume whatever computational model you like, and assume that the graph is represented however you like.

[10] 6. Problem 0.11 (3rd edition of Sipser).

Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = \big(S(n)\big)^2$ for every $n \geq 1$.

[5] a. $S(n) = n(n+1)/2$.

[5] b. $C(n) = (n^4 + 2n^3 + n^2)/4 = n^2(n+1)^2/4$.