CPSC 421/501 Intro to Theory of Computing (Term 1, 2013-14)
Assignment 6

Due: Wednesday November 13th, in class.

Question 1: [10 marks]

Let G be a social network (i.e., a graph) whose vertices correspond to people and whose (undirected) edges correspond to friend relationships between pairs of people.

Suppose that some pairs of people (who are friends) get married. We allow the possibility of polygamy: person a could simultaneously be married to person b and to person c. A friend relationship between person x and person y is said to be **defunct** if x did not marry y, but either x or y married someone else say, person z. (For example, if x and y are friends, but x marries z, then x and y are unlikely to hang out anymore.) The entire social network is said to be **dreary** if every relationship is either a married couple, or a defunct friendship.

Let $DREARY$ the the language consisting of strings $\langle G, k \rangle$ such that G is a social network in which k marriages can make the network become dreary. Prove that $DREARY$ is NP-complete.

Question 2: [10 marks]

Let $G = (V, E)$ be a directed graph. A **circuit** C is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ with $k \geq 3$ such that $v_k v_1 \in E$ and $v_i v_{i+1} \in E$ for all $i = 1, \ldots, k-1$. A **partition into circuits** is a partition of V into $C_1, \ldots, C_p$ such that each C_i is a circuit. Every vertex must be contained in exactly one of the circuits. The number of circuits is irrelevant and the length of each circuit is irrelevant (as long as each circuit has at least 3 vertices).

Define

$$PIC = \{ \langle G \rangle : G \text{ is a directed graph with a partition into circuits} \}.$$

Prove that PIC is NP-complete.

Question 3: [10 marks]

Recall the problem

$$CLIQUE = \{ \langle G, k \rangle : G \text{ has a clique of size } k \}.$$

We showed in Lecture 21 that $CLIQUE \in NP$ and in Lecture 23 that $CLIQUE$ is NP-complete. So, if $P = NP$ then $CLIQUE \in P$.

Consider instead the problem

$$MAXCLIQUE = \{ \langle G, k \rangle : \text{the largest clique in } G \text{ has exactly } k \text{ vertices} \}.$$

This problem is *not believed* to be in NP.

- **(a):** Prove that if $P = NP$ then $MAXCLIQUE \in P$.
- **(b):** Prove that if $MAXCLIQUE \in NP$ then $NP = coNP$.

OPTIONAL BONUS QUESTIONS:

Question 4: [1200 marks]

Suppose that we are given a graph $G = (V, E)$. Let $n = |V|$ and $m = |E|$. Suppose that $m = k \cdot (n-1)$ for some $k \in \mathbb{N}$, and that we are given a partition of E into disjoint sets $E_1, \ldots, E_{n-1}$ with $|E_i| = k$ for every i. (We can think of each edge as being assigned one of $n-1$ different colors, such that each color is assigned to exactly k edges.)

A **spanning tree** of G is a set $T \subseteq E$ with $|T| = n-1$ that contains no cycles. A **rainbow spanning tree** is a spanning tree T with $|T \cap E_i| = 1$ for every i. (We can think of tree T as having exactly one edge of each color.)

Consider the computational problem of deciding, given G and $E_1, \ldots, E_{n-1}$ as above, whether there exists a rainbow spanning tree. This problem is known to be in P (by applying a powerful tool known as "matroid intersection").

Consider the computational problem of deciding, given G and $E_1, \ldots, E_{n-1}$ as above, whether E can be *partitioned* into *disjoint* sets $T_1, \ldots, T_k$, where each T_i is a rainbow spanning tree. Prove that this problem is either in P or is NP-complete. Feel free to use any books, or internet resources, or collaborate with any other students, or ask me any questions. To get CPSC 421/501 credit for this question, you must solve it anytime before December 12th.