CPSC 550: Machine Learning II	2008/9 Term 2

Lecture 6 — Feb 5, 2009

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This lecture continues the discussion of the Lebesque integral and introduces the concepts of measurable spaces, measurable functions, and simple functions.

6.1 The Lebesgue integral continued

Unlike the Reimann integral which partitions the domain, the Lebesgue integral partitions the range into intervals $[y_i, y_{i+1})$, $i \in \{1, ..., n\}$. The Lebesgue sum is then defined to be:

$$\sum_{i=1}^{n} y_i^* \nu\left(A_i\right) \tag{6.1}$$

Where $\nu\left(A_i\right)$ is the measure of the set A_i and A_i is defined as:

$$A_i = \{x : y_i \leq f(x) < y_{i+1}\} \tag{6.2}$$

In other words, for each interval $[y_i, y_{i+1})$ we take the measure of the rectangles formed from the intersection of the interval with the function, as shown in figure 6.1, and multiply them by the height of the interval, and repeat this for each partition.

Taking the number of partitions to infinity, and taking them infentesimally small so that $y_i^* \to f(x)$, we get the Lebesgue integral:

$$\int y_i^* d\nu(A_i) \to \int f(x) d\nu(x) = \int f d\nu \tag{6.3}$$

Example 6.1. *We return to the problem of integrating over a particular function from last class, defined as:*

$$f(x) = I_{\mathbb{Q}}(x) = \left\{ \begin{array}{ll} 1 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{array} \right\}$$

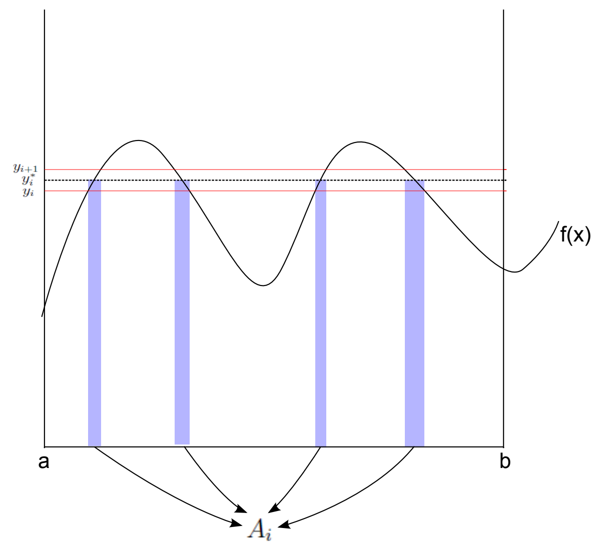


Figure 6.1. The rectangles formed by intersecting a function $f(x)$ with a particular interval $[y_i, y_{i+1})$. We assign each of these rectangles a "height" y_i^*. The Lebesgue sum is the sum of the area of these rectangles over all intervals.

If you recall, this was not Reimann integrable, however using the Lebesgue integral we are able to find that

$$\int I_{\mathbb{Q}}(x) d\nu\left(x\right) = \nu\left(\mathbb{Q} \cap \mathbb{R}\right) = 0$$

The details of the solution are beyond the scope of this course, but can be found in [1].

6.2 Measurable Spaces

Now we will move on to the notion of measurability. First we will need to define the notion of a measurable space:

Definition 6.2. *A* ***measurable space*** *consists of a set of events Ω, and a collection of subsets $\mathfrak{F}$ of Ω such that:*

(i) The null space is in $\mathfrak{F}$: $\phi \in \mathfrak{F}$

(ii) If the set A is in $\mathfrak{F}$ then so is its complement: $A \in \mathfrak{F} \implies \overline{A} \in \mathfrak{F}$

(iii) $\mathfrak{F}$ *is closed under countable unions:* $A_i \in \mathfrak{F}\ \forall i \implies \bigcup_i A_i \in \mathfrak{F}$

Under these properties, we can say that $\mathfrak{F}$ *consists of measurable events.* $\mathfrak{F}$ *is known as a* ***σ-field***.

Once we have a measurable space, we will want to define the concept of a measure on that space.

Definition 6.3. *Given a measurable space* $(\Omega, \mathfrak{F})$*, a measure* $\nu : \mathfrak{F} \to \mathbb{R} \cup \{+\infty, -\infty\}$ *is a function that assigns a real nuber to events in* $\mathfrak{F}$*. To be considered a measure,* ν *must have the following properties [3]:*

(i) $\nu(E) > 0\ \forall\ E \in \mathfrak{F}$

(ii) $\nu(\phi) = 0$

(iii) if $\{E_i\}_{i \in I}$ *is a set of pairwise disjoint sets from* $I \in \mathfrak{F}$ *then*
$\nu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \nu\left(E_i\right)$

Definition 6.4. *The* ***Borel σ-field*** *on the set of real numbers* $\mathbb{R}$*, denoted* $\mathcal{B}(\mathbb{R})$*, is the smallest σ-field of* $\mathbb{R}$ *that contains all intervals. The* ***Borel measure*** *is a measure* ν *defined on the Borel σ-field*

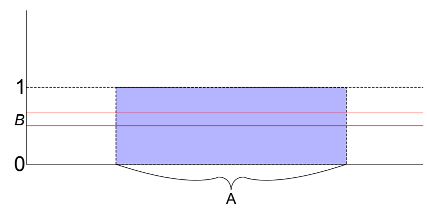
6.3 Measurable Functions

Now that we have the concept of measurable spaces, we can discuss measurable functions which transform points from one measurable space to another measurable space.

Definition 6.5. *Let* $(\Omega, \mathfrak{F})$ *and* $(\Sigma, \mathfrak{T})$ *be two measurable spaces, then* $f : \Omega \to \Sigma$ *is measurable if for any subset* $E \in \mathfrak{T}$*,* $f^{-1}(E) = \{x : f(x) \in E\} \in \mathfrak{F}$*.* $f^{-1}(E)$ *is called the* ***preimage*** *of* E *under* f *[2].*

Example 6.6. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *on the measurable spaces* $(\mathbb{R}, \mathcal{B}(\mathbb{R})), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$*:*

$$f(x) = I_A(x) = \left\{ \begin{array}{ll} 1 & \text{if x} \in \text{A} \\ 0 & \text{otherwise} \end{array} \right\}$$



*In the diagram above, B is an interval over the range, known as a **Borel set** since it is a subset of $\mathcal{B}(\mathbb{R})$. We can determine the $I_A^{-1}(B)$ by looking at the values of the domain which map to any interval B. There are 4 possible cases:*

$$I_A^{-1}(B) = \begin{cases} \phi & \text{if } 0 \notin B,\, 1 \notin B \\ A & \text{if } 0 \notin B,\, 1 \in B \\ \overline{A} & \text{if } 0 \in B,\, 1 \notin B \\ \mathbb{R} & \text{if } 0 \in B,\, 1 \in B \end{cases}$$

In this case, we see that the preimage $I_A^{-1}(B)$ is an element of $\mathcal{B}(\mathbb{R})$ for all possible choices of B. Thus, $I_A(X)$ is measurable.

6.4 Simple Functions

Now that we have the theory of a measurable function, we can define the Lebesgue integral more formally through the use of simple functions.

First, we will need to define the integral of an indicator function of a measurable set S:

$$\int I_S d\nu \quad = \quad \nu(S) \tag{6.4}$$

Definition 6.7. *Let $(\Omega, \mathfrak{F})$ be a measurable space. Let $A_1, ..., A_n \in \mathfrak{F}$ be a sequence of measurable sets, and let $a_1, ..., a_n$ be a sequence of non-negative real numbers. A **simple function** is a function of the form[4]:*

$$f(x) \quad = \quad \sum_{i=1}^{n} a_i I_{A_i}(x) \tag{6.5}$$

Where $I_{A_i}(x)$ is 1 if $x \in A_i$ and 0 otherwise.

Using this definition, we can build the Lebesgue integral for non-negative simple functions by taking:

$$\int f d\nu = \int \sum_{i=1}^{n} a_i I_{A_i}(x) d\nu(x) = \sum_{i=1}^{n} a_i \int I_{A_i}(x) d\nu(x) = \sum_{i=1}^{n} a_i \nu(A_i) \quad (6.6)$$

This can be extended to define the integral of a real-valued simple function, and furthermore it can be shown that a measurable function is the pointwise limit of a sequence of simple functions. Thus, using the Lebesgue integral over measurable simple functions, it is possible to compute the Lebesgue integral over arbitrary, measurable functions. More information can be found in [5].

Exercise 6.8. *Show that:*

(i) $\sum_{i=1}^{n} a_i I_{A_i}(x)$

(ii) $\lim_{n\to\infty} \sum_{i=1}^{n} a_i I_{A_i}(x)$

are measurable.

6.5 Measure Continuity

In the next class we will cover measure continuity, where we will prove the following proposition:

Proposition 6.9. *Let* $A_1 \subset A_2 \subset A_3 \subset ...$

Then $\nu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \nu(A_n)$
where $A_n = \bigcup_{i=1}^{n} A_i$

Bibliography

[1] J.H. Lifton. Measure Theory and Lebesgue Integration.

[2] Feldman, Joel. Review of Measurable Functions. http://www.math.ubc.ca/~feldman/m420/mblefnsReview.pdf.

[3] Wikipedia. Wikipedia article on measure. http://en.wikipedia.org/wiki/Measure_(mathematics).

[4] Wikipedia. Wikipedia article on simple functions. http://en.wikipedia.org/wiki/Simple_function.

[5] Wikipedia. Wikipedia article on the Lebesgue integral. http://en.wikipedia.org/wiki/Lebesgue_integral.