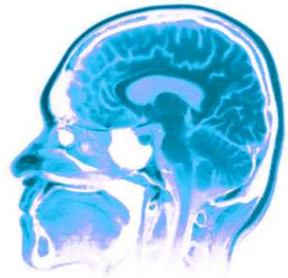
CPSC540



Discrete Probability and Bayesian Learning

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September, 2011
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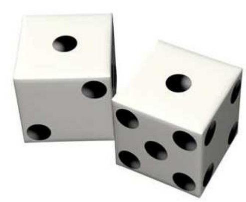
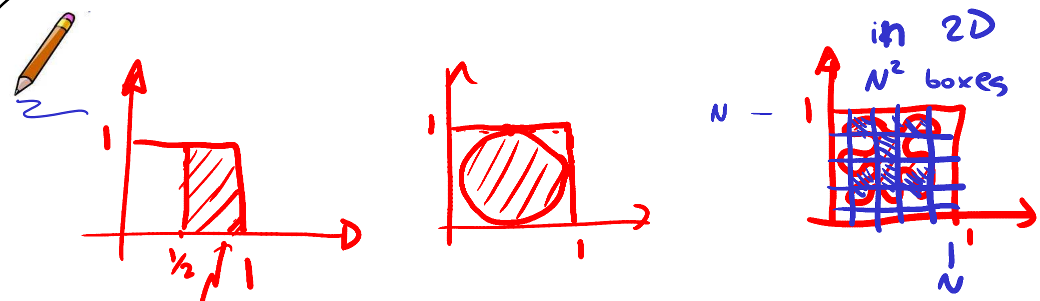
Probability



Probability theory is the formal study of the laws of chance. It is our tool for dealing with uncertainty. Notation:

- **Sample space:** is the set Ω of all outcomes of an experiment.
- **Outcome:** what we observed. We use $\omega \in \Omega$ to denote a particular outcome. *e.g.* for a die we have $\Omega = \{1, 2, 3, 4, 5, 6\}$ and ω could be any of these six numbers.
- **Event:** is a subset of Ω that is well defined (measurable). *e.g.* the event $A = \{even\}$ if $w \in \{2, 4, 6\}$

Frequentist interpretation



$P(\blacksquare) = \frac{1}{2}$

$P(\boxed{/}) = \frac{\pi (1/2)^2}{1} = \frac{\pi}{4}$

$P(\boxed{/}) = \frac{\# \text{ squares shaded}}{\# \text{ squares}}$

make squares smaller for more precision!

Axiomatic interpretation

The axiomatic view is a more elegant mathematical solution. Here, a **probabilistic model** consists of the triple $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the sigma-field (collection of measurable events) and P is a function mapping $\mathcal{F}$ to the interval $[0, 1]$. That is, with each event $A \in \mathcal{F}$ we associate a probability $P(A)$.

$$\begin{cases} \Omega = \{1, 2, 3, 4, 5, 6\} \\ \mathcal{F} = \text{Powerset} = \{\emptyset, 1, 2 \ldots, 6, \{1,2\}, \ldots\} \\ P(\text{even}) = 1/2 \\ P(\text{odd}) = 1/2 \end{cases}$$

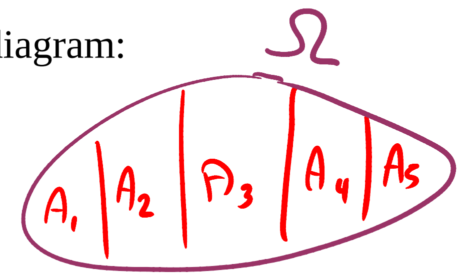
The axioms

1. $P(\emptyset) = 0 \leq p(A) \leq 1 = P(\Omega)$

2. For **disjoint sets** A_n, $n \geq 1$, we have

$$P\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

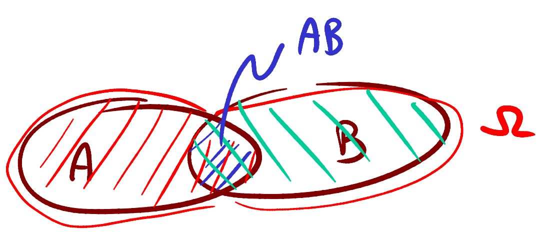
Venn diagram:

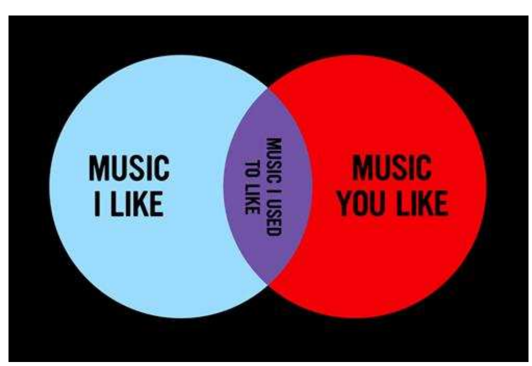


$P(\Omega) = P(A_1) + P(A_2) + \cdots + P(A_5)$

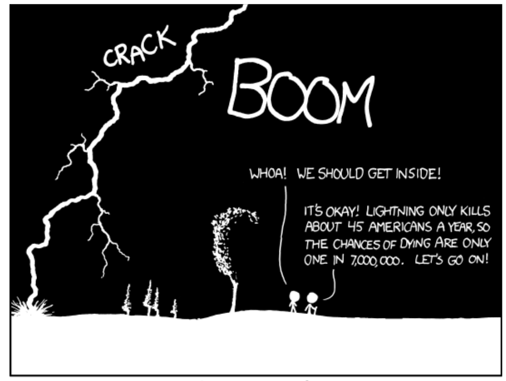
OR and AND operations

$$P(A \overset{\text{OR}}{+} B) = P(A) + P(B) - P(\overset{\text{and}}{AB})$$





Conditional probability



THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

given, and

$$P(A|B) \triangleq \frac{P(AB)}{P(B)}$$

where $P(A|B)$ is the **conditional probability** of A given that B occurs, $P(B)$ is the **marginal probability** of B and $P(AB)$ is the **joint probability** of A and B. In general, we obtain a **chain rule**

$$P(A_{1:n}) = P(A_n|A_{1:n-1})P(A_{n-1}|A_{1:n-2}) \ldots P(A_2|A_1)P(A_1)$$

If the events A and B are **independent**, we have $P(AB) = P(A)P(B)$. $= P(A|\cancel{B})P(B)$ $P(A)$

Conditional probability example

⋆ Assume we have an urn with 3 red balls and 1 blue ball: $U = \{r, r, r, b\}$. What is the probability of drawing (without replacement) 2 red balls in the first 2 tries?

$$P(d_1 = r) = \frac{3}{4}$$

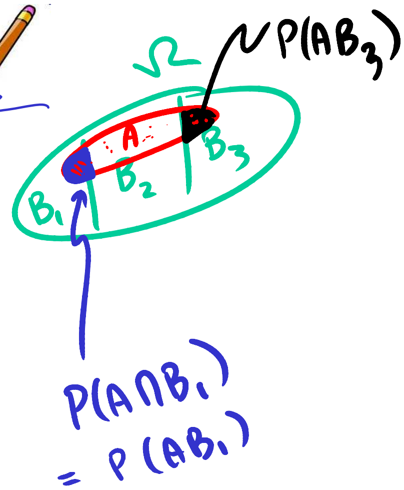
$$P(d_2 = r, d_1 = r) = P(d_2 = r | d_1 = r)\, P(d_1 = r)$$

$$= \frac{2}{3}\left(\frac{3}{4}\right) = \frac{1}{2}$$

Marginalization

Let the sets $B_{1:n}$ be disjoint and $\bigcup_{i=1}^{n} B_i = \Omega$. Then

$$\underline{P(A)} = \sum_{i=1}^{n} P(A, B_i)$$



$$P(A) = P(A \cap \Omega)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$

$$P(A) = P(AB_1) + P(AB_2) + P(AB_3)$$

Marginalization example

⋆ What is the probability that the second ball drawn from our urn will be red?

$$P(d_2 = r) = \sum_{d_1 \in \{b, r\}} P(d_2 = r, d_1)$$

$$= \sum_{d_1} P(d_2 = r \mid d_1) P(d_1)$$

$$= P(d_2 = r \mid d_1 = r) P(d_1 = r) + P(d_2 = r \mid d_1 = b) P(d_1 = b)$$

Bayes rule

Bayes rule allows us to reverse probabilities:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

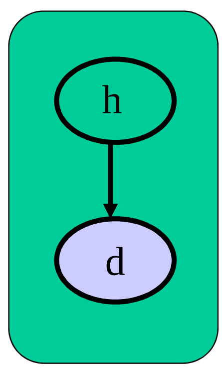
$$P(AB) = P(B|A)P(A) = P(A|B)P(B)$$

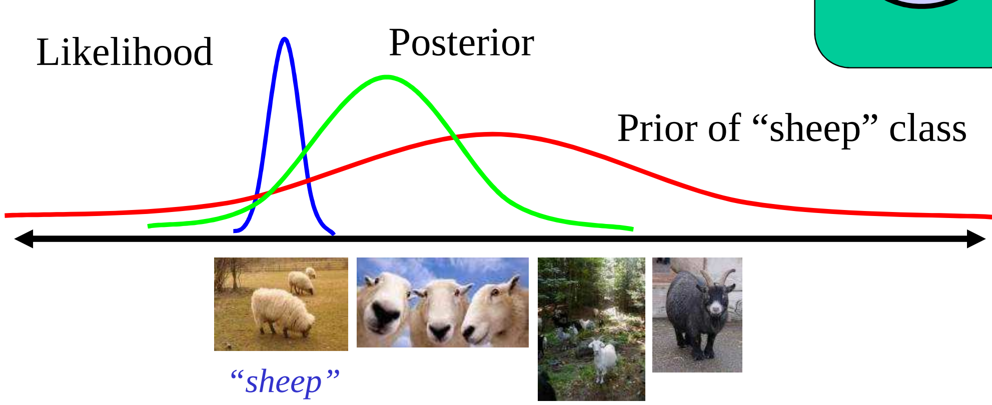
$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$= \frac{P(A|B)P(B)}{\sum_{B} P(A|B)P(B)}$$

Learning and Bayesian inference

$$p(h|d) = \frac{p(d|h)p(h)}{\sum_{h' \in H} p(d|h')p(h')}$$





Speech recognition

P(words | sound) α P(sound | words) P(words)

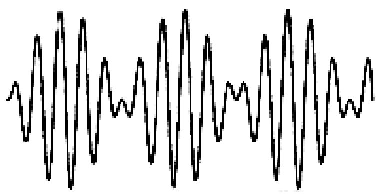
Final beliefs

Likelihood of data
eg mixture of Gaussians

Language model
eg Markov model

Hidden Markov Model (HMM)

"Recognize speech"



"Wreck a nice beach"

Definition of discrete r.v.s

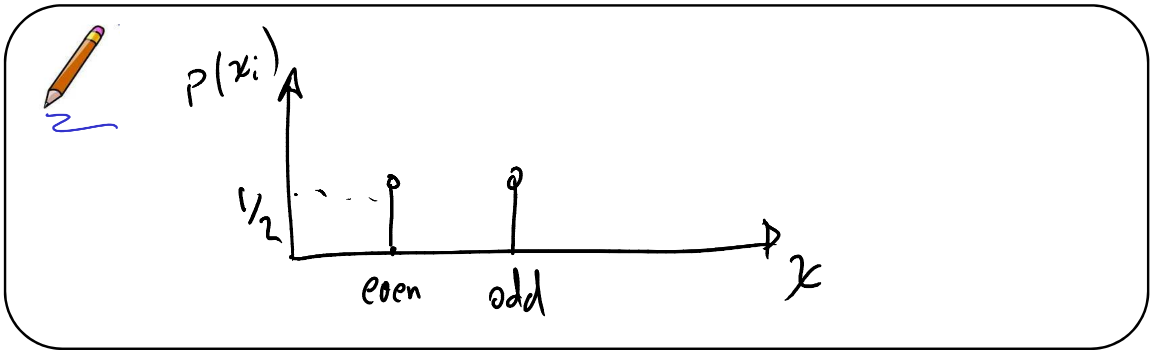
Let E be a discrete set, e.g. $E = \{0, 1\}$. A **discrete random variable** (r.v.) is a map from Ω to E:

$$X(w) : \Omega \mapsto E$$

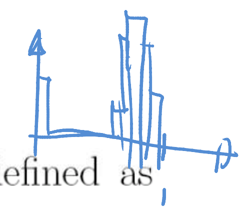
such that for all $x \in E$ we have $\{w | X(w) \leq x\} \in \mathcal{F}$. Since $\mathcal{F}$ denotes the measurable sets, this condition simply says that we can compute (measure) the probability $P(X = x)$.

Probability distributions

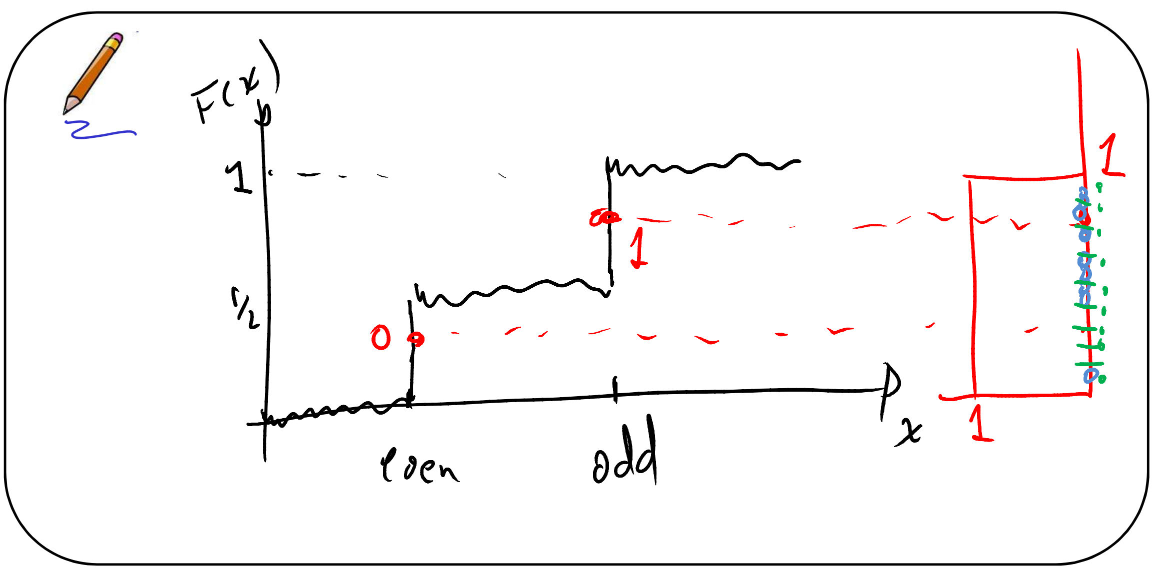
⋆ Assume we are throwing a die and are interested in the events $E = \{even, odd\}$. Here $\Omega = \{1, 2, 3, 4, 5, 6\}$. The r.v. takes the value $X(w) = even$ if $w \in \{2, 4, 6\}$ and $X(w) = odd$ if $w \in \{1, 3, 5\}$. We describe this r.v. with a **probability distribution** $p(x_i) = P(X = x_i) = \frac{1}{2}$, $i = 1, \ldots, 2$



The CDF



The **cumulative distribution function** is defined as $F(x) = P(X \leq x)$ and would for this example be:



Expectation

The expectation of a discrete random variable X is

$$\mathbb{E}[X] = \sum_{E} x_i p(x_i)$$

The expectation operator is linear, so $\mathbb{E}(ax_1+bx_2) = a\mathbb{E}(x_1)+b\mathbb{E}(x_2)$. In general, the expectation of a function $f(X)$ is

$$\mathbb{E}[f(X)] = \sum_{E} f(x_i)\, p(x_i)$$

Mean: $\mu \triangleq \mathbb{E}(X)$

Variance: $\sigma^2 \triangleq \mathbb{E}[(X-\mu)^2]$

Bernoulli r.v.s and the indicator function

Let $E = \{0, 1\}$, $P(X = 1) = \lambda$, and $P(X = 0) = 1 - \lambda$.

We now introduce the *set indicator variable*. (This is a very useful notation.)

$$\mathbb{I}_A(w) = \begin{cases} 1 \;\; if & w \in A; \\ 0 \;\; otherwise. & \end{cases}$$

Using this convention, the probability distribution of a **Bernoulli** random variable reads:

$$p(x) = \lambda^{\mathbb{I}_{\{1\}}(x)}(1-\lambda)^{\mathbb{I}_{\{0\}}(x)}.$$

if $x=1$ $\mathbb{I}_{\{1\}}(x)=1$ $P(X=1)=\lambda$

For **identical and independent distributed** (i.i.d.) data:

$$x_i \sim \underbrace{\theta^{\mathbb{I}_1(x_i)} (1-\theta)^{\mathbb{I}_0(x_i)}}_{P(x_i|\theta)}$$

$$p(x_{1:n}|\theta) = \prod_{i=1}^{n} P(x_i|\theta)$$

$$\mathcal{L}(\theta) = \log p(x_{1:n}|\theta) = \sum_{i=1}^{n} \mathrm{Log}\, P(x_i|\theta)$$

Maximum likelihood example

Let $x_{1:n}$, with $x_i \in \{0, 1\}$, be i.i.d. Bernoulli:

$$p(x_{1:n}|\theta) = \prod_{i=1}^{n} p(x_i|\theta)$$

$$= \prod_{i=1}^{n} \theta^{\mathbb{I}_1(x_i)} (1-\theta)^{\mathbb{I}_0(x_i)}$$

$$= \theta^{\sum_{i=1}^{n} \mathbb{I}_1(x_i)} (1-\theta)^{\sum_{i=1}^{n} \mathbb{I}_0(x_i)}$$

$$= \theta^{m} (1-\theta)^{n-m}$$

$m \equiv$ # of 1's

$n-m =$ # of 0's

Maximum likelihood example

With $m \triangleq \sum x_i$, we have

$$\mathcal{L}(\theta) = \log P(x_{1:n}|\theta)$$

$$\mathcal{L}(\theta) = m \log \theta + (n-m)\log(1-\theta)$$

Differentiating, we get

$$\frac{\partial \mathcal{L}(\theta)}{\partial \theta} = \frac{m}{\theta} + (n-m)\frac{1}{1-\theta}(-1)$$

$$= \frac{m}{\theta} - \frac{n-m}{1-\theta} \rightarrow 0$$

$$\boxed{\theta = \frac{m}{n}}$$

Bayesian learning

Given our **prior** knowledge $p(\theta)$ and the data **model** $p(\cdot|\theta)$, the Bayesian approach allows us to update our prior using the new data $x_{1:n}$ as follows:

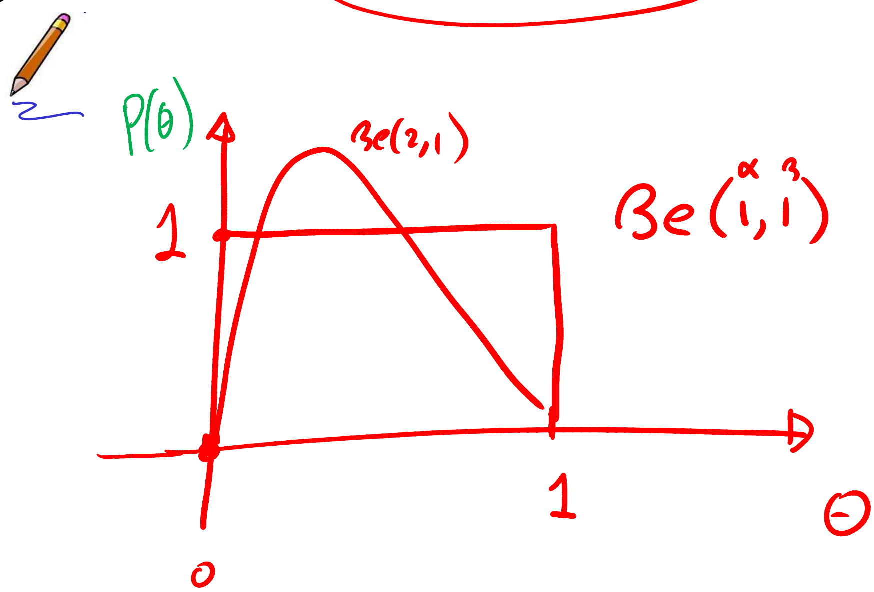
posterior

$$p(\theta|x_{1:n}) = \frac{\overbrace{p(x_{1:n}|\theta)}^{\text{lik}}\overbrace{p(\theta)}^{\text{prior}}}{p(x_{1:n})}$$

where $p(\theta|x_{1:n})$ is the **posterior distribution**, $p(x_{1:n}|\theta)$ is the likelihood and $p(x_{1:n})$ is the **marginal likelihood** (evidence). Note

$$p(x_{1:n}) = \int p(x_{1:n}|\theta)p(\theta)d\theta$$

Beta prior



$$\text{mean}\left[Be(\alpha,\beta)\right] = \frac{\alpha}{\alpha+\beta}$$

Example

Let $x_{1:n}$, with $x_i \in \{0,1\}$, be i.i.d. Bernoulli: $x_i \sim \mathcal{B}(1,\theta)$

$$p(x_{1:n}|\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \theta^m (1-\theta)^{n-m}$$

Let us choose the following **Beta** prior distribution:

$$p(\theta) = \int \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} d\theta = 1$$

$$\int \theta^{\alpha-1}(1-\theta)^{\beta-1} d\theta = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

where Γ denotes the Gamma-function. For the time being, α and β are fixed **hyper-parameters**. The posterior distribution is proportional to:

$$p(\theta|x_{1:n}) \propto P(x_{1:n}|\theta) \quad P(\theta)$$

$$= \theta^{m}(1-\theta)^{n-m} \quad \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \theta^{m+\alpha-1}(1-\theta)^{n-m+\beta-1}$$

with normalisation constant

$$P(\theta|x_{1:n}) = \frac{\Gamma(m+\alpha)\,\Gamma(n-m+\beta)}{\Gamma(n+\alpha+\beta)}$$

$$\times \; \theta^{m+\alpha-1}(1-\theta)^{n-m+\beta-1}$$